ON THE GENERALIZED COMMUTING VARIETIES OF A REDUCTIVE LIE ALGEBRA.

JEAN-YVES CHARBONNEL AND MOUCHIRA ZAITER

ABSTRACT. The generalized commuting and isospectral commuting varieties of a reductive Lie algebra have been introduced in a preceding article. In this note, it is proved that their normalizations are Gorenstein with rational singularities. Moreover, their canonical modules are free of rank 1. In particular, the usual commuting variety is Gorenstein with rational singularities and its canonical module is free of rank 1.

CONTENTS

1. INTRODUCTION

In this note, the base field $\Bbbk$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, ℓ is its rank, $\dim \mathfrak{g} = \ell + 2n$ and G is its adjoint group. As usual, $\mathfrak{b}$ denotes a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$, and B the normalizer of $\mathfrak{b}$ in G.

1.1. **Main results.** By definition, for $k \geqslant 1$, the generalized commuting variety $\mathcal{C}^{(k)}$ is the closure in $\mathfrak{g}^k$ of the set of elements whose components are in a same Cartan subalgebra. Denoting by $\mathcal{B}^{(k)}$ the subset of elements of $\mathfrak{g}^k$ whose components are in a same Borel subalgebra and by $\mathcal{B}_{\mathrm{n}}^{(k)}$ its normalization, the generalized isospectral commuting variety $\mathcal{C}_{\mathrm{x}}^{(k)}$ is above $\mathcal{C}^{(k)}$ and under the inverse image of $\mathcal{C}^{(k)}$ in $\mathcal{B}_{\mathrm{n}}^{(k)}$. For $k = 2$, $\mathcal{C}^{(2)}$ is the commuting variety of $\mathfrak{g}$ and $\mathcal{C}_{\mathrm{x}}^{(2)}$ is the isospectral commuting variety considered by V. Ginzburg in [Gi12]. According to [CZ14, Proposition 5.6], $\mathcal{C}_{\mathrm{x}}^{(k)}$ is an irreducible variety. For studying these varieties, it is very useful to consider the closure in the grassmannian $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under the action of B in $\mathrm{Gr}_\ell(\mathfrak{g})$. Denoting by X this variety, $G.X$ is the closure of the orbit of $\mathfrak{h}$ under G. Let $\mathcal{E}_0$ and $\mathcal{E}$ be the restrictions to X and $G.X$ of the tautological vector bundle over $\mathrm{Gr}_\ell(\mathfrak{g})$ respectively. Denoting by

Date: August 26, 2015.
1991 *Mathematics Subject Classification.* 14A10, 14L17, 22E20, 22E46 .
Key words and phrases. polynomial algebra, complex, commuting variety, desingularization, Gorenstein, Cohen-Macaulay, rational singularities, cohomology.

$\mathcal{E}^{(k)}$ the fiber product over $G.X$ of k copies of $\mathcal{E}$, $\mathcal{E}^{(k)}$ is a subbundle of $G.X \times \mathfrak{g}^k$ and $\mathcal{C}^{(k)}$ is the image of $\mathcal{E}^{(k)}$ by the canonical projection $G.X \times \mathfrak{g}^k \longrightarrow \mathfrak{g}^k$. Analogously, denoting by $\mathcal{E}_0^{(k)}$ the restriction of $\mathcal{E}^{(k)}$ to X, the image $\mathfrak{X}_{0,k}$ of $\mathcal{E}_0^{(k)}$ by the projection $X \times \mathfrak{g}^k \longrightarrow \mathfrak{g}^k$ is the closure in $\mathfrak{b}^k$ of the set of elements whose components are in a same Cartan subalgebra. The fiber bundle $G \times_B \mathcal{E}_0^{(k)}$ is a vector bundle of rank ℓ over the fiber bundle $G \times_B X$ over G/B. As for $\mathcal{C}^{(k)}$, there is a surjective morphism from $G \times_B \mathcal{E}_0^{(k)}$ onto $\mathcal{C}_{\mathrm{x}}^{(k)}$. As a matter of fact, the three morphisms:

$$\mathcal{E}_0^{(k)} \xrightarrow{\tau_{0,k}} \mathfrak{X}_{0,k}\ , \qquad \mathcal{E}^{(k)} \xrightarrow{\tau_k} \mathcal{C}^{(k)}\ , \qquad G \times_B \mathcal{E}_0^{(k)} \xrightarrow{\tau_{*,k}} \mathcal{C}_{\mathrm{x}}^{(k)}$$

are projective and birational. According to [CZ14, Theorem 1.2], $G.X$ is smooth in codimension 1 so that so is $\mathcal{E}^{(k)}$. By [C15, Theorem 1.1], X is normal and Gorenstein then so are $\mathcal{E}_0^{(k)}$ and $G \times_B \mathcal{E}_0^{(k)}$. Denoting by $(G.X)_{\mathrm{n}}$ the normalization of $G.X$, the pullback bundle of $\mathcal{E}^{(k)}$ over $(G.X)_{\mathrm{n}}$ is the normalization of $\mathcal{E}^{(k)}$. Denoting it by $\mathcal{E}_{\mathrm{n}}^{(k)}$ we have projective birational morphisms:

$$\mathcal{E}_0^{(k)} \xrightarrow{\tau_{\mathrm{n},0,k}} \widetilde{\mathfrak{X}_{0,k}}\ , \qquad \mathcal{E}_{\mathrm{n}}^{(k)} \xrightarrow{\tau_{\mathrm{n},k}} \widetilde{\mathcal{C}^{(k)}}\ , \qquad G \times_B \mathcal{E}_0^{(k)} \xrightarrow{\tau_{\mathrm{n},*,k}} \widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}\ ,$$

with $\widetilde{\mathfrak{X}_{0,k}}$, $\widetilde{\mathcal{C}^{(k)}}$, $\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}$ the normalizations of $\mathfrak{X}_{0,k}$, $\mathcal{C}^{(k)}$, $\mathcal{C}_{\mathrm{x}}^{(k)}$ respectively. According to [C15, Proposition 4.6], for some smooth big open subset O_0 of $\mathfrak{X}_{0,k}$, there exists a regular differential form of top degree without zero. Moreover, the restriction of $\tau_{0,k}$ to $\tau_{0,k}^{-1}(O_0)$ is an isomorphism onto O_0. By a simple argument, $\mathcal{C}^{(k)}$ and $\mathcal{C}_{\mathrm{x}}^{(k)}$ are smooth in codimension 1. Moreover, for some smooth big open subsets O and O_* in $\mathcal{C}^{(k)}$ and $\mathcal{C}_{\mathrm{x}}^{(k)}$ respectively, the restrictions of τ_k and $\tau_{*,k}$ to $\tau_k^{-1}(O)$ and $\tau_{*,k}^{-1}(O_*)$ are isomorphisms onto O and O_* respectively. The main observation of this note is that there are regular differential forms of top degree on O and O_* without zero. As a result, we have the following theorem:

Theorem 1.1. *The varieties $\widetilde{\mathfrak{X}_{0,k}}$, $\widetilde{\mathcal{C}^{(k)}}$, $\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}$ are Gorenstein with rational singularities and their canonical modules are free of rank* 1. *Moreover, $(G.X)_{\mathrm{n}}$ is Gorenstein with rational singularities.*

In particular, we give a new proof of a Ginzburg's result [Theorem 1.3.4][Gi12]. For $k = 2$, $\mathcal{C}^{(2)}$ is the commuting variety of $\mathfrak{g}$ by [Ri79] and it is normal by [C12, Theorem 1.1]. So the commuting variety of $\mathfrak{g}$ is Gorenstein with rational singularities and its canonical module is free of rank 1. Since $\widetilde{\mathfrak{X}_{0,k}}$ has rational singularities, we get that some cohomological groups in positive degree are equal to 0 and we deduce that $\mathfrak{X}_{0,k}$ is normal.

This note is organized as follows. In Section 2, the variety $\mathcal{X}$ is introduced and we prove that on the smooth loci of $\mathcal{X}$ and $G \times_B \mathfrak{b}$, there are regular differential forms of top degree without zero. In Section 3, we recall some results about $\mathcal{E}$, X, $G.X$, $(G.X)_{\mathrm{n}}$. In Section 4, we give some results about $\mathcal{C}^{(k)}$ and $\mathcal{C}_{\mathrm{x}}^{(k)}$ and we prove the main result about regular differential forms of top degree on the smooth loci of these varieties. As a result, we get the main result of the note in Section 5. The goal of Section 6 is the normality of $\mathfrak{X}_{0,k}$. At last, in the appendix, some results are given to prove the normality of $\mathfrak{X}_{0,k}$ and Theorem 1.1.

1.2. **Notations.** • An algebraic variety is a reduced scheme over $\Bbbk$ of finite type.

• For V a vector space, its dual is denoted by V^* and the augmentation ideal of its symmetric algebra $\mathrm{S}(V)$ is denoted by $\mathrm{S}_+(V)$. For A a graded algebra over $\mathbb{N}$, A_+ is the ideal generated by the homogeneous elements of positive degree.

• All topological terms refer to the Zariski topology. If Y is a subset of a topological space X, denote by $\overline{Y}$ the closure of Y in X. For Y an open subset of the algebraic variety X, Y is called *a big open subset* if the codimension of $X \setminus Y$ in X is at least 2. For Y a closed subset of an algebraic variety X, its dimension

is the biggest dimension of its irreducible components and its codimension in X is the smallest codimension in X of its irreducible components. For X an algebraic variety, $\mathcal{O}_X$ is its structural sheaf, $X_{\rm sm}$ is its smooth locus, $\Bbbk[X]$ is the algebra of regular functions on X and $\Bbbk(X)$ is the field of rational functions on X when X is irreducible. When X is smooth and irreducible, the sheaf of regular differential forms of top degree on X is denoted by Ω_X.

• For X an algebraic variety and for $\mathcal{M}$ a sheaf on X, $\Gamma(V,\mathcal{M})$ is the space of local sections of $\mathcal{M}$ over the open subset V of X. For i a nonnegative integer, $\mathrm{H}^i(X,\mathcal{M})$ is the i-th group of cohomology of $\mathcal{M}$. For example, $\mathrm{H}^0(X,\mathcal{M}) = \Gamma(X,\mathcal{M})$.

Lemma 1.2. [EGAII, Corollaire 5.4.3] *Let X be an irreducible affine algebraic variety and let Y be a desingularization of X. Then $\mathrm{H}^0(Y,\mathcal{O}_Y)$ is the integral closure of $\Bbbk[X]$ in its fraction field.*

• For E a set and k a positive integer, E^k denotes its k-th cartesian power. If E is finite, its cardinality is denoted by $|E|$.

• For $\mathfrak{a}$ reductive Lie algebra, its rank is denoted by $\mathrm{rk}\,\mathfrak{a}$ and the dimension of its Borel subalgebras is denoted by $\mathrm{b}_{\mathfrak{a}}$. In particular, $\dim\mathfrak{a} = 2\mathrm{b}_{\mathfrak{a}} - \mathrm{rk}\,\mathfrak{a}$.

• If E is a subset of a vector space V, denote by $\mathrm{span}(E)$ the vector subspace of V generated by E. The grassmannian of all d-dimensional subspaces of V is denoted by $\mathrm{Gr}_d(V)$. By definition, a *cone* of V is a subset of V invariant under the natural action of $\Bbbk^* := \Bbbk\setminus\{0\}$ and a *multicone* of V^k is a subset of V^k invariant under the natural action of $(\Bbbk^*)^k$ on V^k.

• The dual of $\mathfrak{g}$ is denoted by $\mathfrak{g}^*$ and it identifies with $\mathfrak{g}$ by a given non degenerate, invariant, symmetric bilinear form $\langle .\,,.\rangle$ on $\mathfrak{g}\times\mathfrak{g}$, extending the Killing form of $[\mathfrak{g},\mathfrak{g}]$.

• Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$. Denote by $\mathcal{R}$ the root system of $\mathfrak{h}$ in $\mathfrak{g}$ and by $\mathcal{R}_+$ the positive root system of $\mathcal{R}$ defined by $\mathfrak{b}$. The Weyl group of $\mathcal{R}$ is denoted by $W(\mathcal{R})$ and the basis of $\mathcal{R}_+$ is denoted by Π. The neutral elements of G and $W(\mathcal{R})$ are denoted by $1_{\mathfrak{g}}$ and $1_{\mathfrak{h}}$ respectively. For α in $\mathcal{R}$, the corresponding root subspace is denoted by $\mathfrak{g}^{\alpha}$ and a generator x_α of $\mathfrak{g}^{\alpha}$ is chosen so that $\langle x_\alpha\,, x_{-\alpha}\rangle = 1$ for all α in $\mathcal{R}$. Let H_α be the coroot of α.

• The normalizers of $\mathfrak{b}$ and $\mathfrak{h}$ in G are denoted by B and $N_G(\mathfrak{h})$ respectively. For x in $\mathfrak{b}$, $\overline{x}$ is the element of $\mathfrak{h}$ such that $x-\overline{x}$ is in the nilpotent radical $\mathfrak{u}$ of $\mathfrak{b}$.

• For X an algebraic B-variety, denote by $G\times_B X$ the quotient of $G\times X$ under the right action of B given by $(g,x).b := (gb, b^{-1}.x)$. More generally, for k positive integer and for X an algebraic B^k-variety, denote by $G^k\times_{B^k} X$ the quotient of $G^k\times X$ under the right action of B^k given by $(g,x).b := (gb, b^{-1}.x)$ with g and b in G^k and B^k respectively.

Lemma 1.3. *Let P and Q be parabolic subgroups of G such that P is contained in Q. Let X be a Q-variety and let Y be a closed subset of X, invariant under P. Then $Q.Y$ is a closed subset of X. Moreover, the canonical map from $Q\times_P Y$ to $Q.Y$ is a projective morphism.*

Proof. Since P and Q are parabolic subgroups of G and since P is contained in Q, Q/P is a projective variety. Denote by $Q\times_P X$ and $Q\times_P Y$ the quotients of $Q\times X$ and $Q\times Y$ under the right action of P given by $(g,x).p := (gp, p^{-1}.x)$. Let $g\mapsto\overline{g}$ be the quotient map from Q to Q/P. Since X is a Q-variety, the map

$$Q\times X \longrightarrow Q/P\times X \qquad (g,x)\longmapsto(\overline{g}, g.x)$$

defines through the quotient an isomorphism from $Q\times_P X$ to $Q/P\times X$. Since Y is a P-invariant closed subset of X, $Q\times_P Y$ is a closed subset of $Q\times_P X$ and its image by the above isomorphism equals $Q/P\times Q.Y$. Hence

$Q.Y$ is a closed subset of X since Q/P is a projective variety. From the commutative diagram:

$$\begin{tikzcd} Q\times_P Y \arrow[r] \arrow[dr] & Q/P\times Q.Y \arrow[d] \\ & Q.Y \end{tikzcd}$$

we deduce that the map $Q\times_P Y \longrightarrow Q.Y$ is a projective morphism. $\square$

• For $k \geqslant 1$ and for the diagonal action of B in $\mathfrak{b}^k$, $\mathfrak{b}^k$ is a B-variety. The canonical map from $G\times\mathfrak{b}^k$ to $G\times_B\mathfrak{b}^k$ is denoted by $(g,x_1,\dots,x_k)\mapsto\overline{(g,x_1,\dots,x_k)}$. Let $\mathcal{B}^{(k)}$ be the image of $G\times\mathfrak{b}^k$ by the map $(g,x_1,\dots,x_k)\mapsto(g(x_1),\dots,g(x_k))$ so that $\mathcal{B}^{(k)}$ is a closed subset of $\mathfrak{g}^k$ by Lemma 1.3. Let $\mathcal{B}_{\rm n}^{(k)}$ be the normalization of $\mathcal{B}^{(k)}$ and η the normalization morphism. We have the commutative diagram:

$$\begin{tikzcd} G\times_B\mathfrak{b}^k \arrow[rr, "\gamma_{\rm n}"] \arrow[dr, "\gamma"'] & & \mathcal{B}_{\rm n}^{(k)} \arrow[dl, "\eta_{\rm n}"] \\ & \mathcal{B}^{(k)} & \end{tikzcd}$$

• Let i_k be the injection $(x_1,\dots,x_k)\mapsto\overline{(1_{\mathfrak{g}},x_1,\dots,x_k)}$ from $\mathfrak{b}^k$ to $G\times_B\mathfrak{b}^k$. Then $\iota_k := \gamma\circ i_k$ and $\iota_{{\rm n},k} := \gamma_{\rm n}\circ i_k$ are closed embeddings of $\mathfrak{b}^k$ into $\mathcal{B}^{(k)}$ and $\mathcal{B}_{\rm n}^{(k)}$ respectively. In particular, $\mathcal{B}^{(k)} = G.\iota_k(\mathfrak{b}^k)$ and $\mathcal{B}_{\rm n}^{(k)} = G.\iota_{{\rm n},k}(\mathfrak{b}^k)$.

• Let e be the sum of the x_β's, β in Π, and let h be the element of $\mathfrak{h}\cap[\mathfrak{g},\mathfrak{g}]$ such that $\beta(h)=2$ for all β in Π. Then there exists a unique f in $[\mathfrak{g},\mathfrak{g}]$ such that (e,h,f) is a principal $\mathfrak{sl}_2$-triple. The one parameter subgroup of G generated by ${\rm ad}\,h$ is denoted by $t\mapsto h(t)$. The Borel subalgebra containing f is denoted by $\mathfrak{b}_-$ and its nilpotent radical is denoted by $\mathfrak{u}_-$. Let B_- be the normalizer of $\mathfrak{b}_-$ in G and let U and U_- be the unipotent radicals of B and B_- respectively.

Lemma 1.4. *Let $k\geqslant 2$ be an integer. Let X be an affine variety and set $Y := \mathfrak{b}^k\times X$. Let Z be a closed subset of Y invariant under the action of B given by $g.(v_1,\dots,v_k,x) = (g(v_1),\dots,g(v_k),x)$ with $(g,v_1,\dots,v_k)$ in $B\times\mathfrak{b}^k$ and x in X. Then $Z\cap\mathfrak{h}^k\times X$ is the image of Z by the projection $(v_1,\dots,v_k,x)\mapsto(\overline{v_1},\dots,\overline{v_k},x)$.*

Proof. For all v in $\mathfrak{b}$,

$$\overline{v} = \lim_{t\to 0} h(t)(v)$$

whence the lemma since Z is closed and B-invariant. $\square$

• For $x\in\mathfrak{g}$, let $x_{\rm s}$ and $x_{\rm n}$ be the semisimple and nilpotent components of x in $\mathfrak{g}$. Denote by $\mathfrak{g}^x$ and G^x the centralizers of x in $\mathfrak{g}$ and G respectively. For $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$ and for A a subgroup of G, set:

$$\mathfrak{a}^x := \mathfrak{a}\cap\mathfrak{g}^x \qquad A^x := A\cap G^x \ .$$

The set of regular elements of $\mathfrak{g}$ is

$$\mathfrak{g}_{\rm reg} \ := \ \{x\in\mathfrak{g} \mid \dim\mathfrak{g}^x = \ell\}.$$

Denote by $\mathfrak{g}_{\rm reg,ss}$ the set of regular semisimple elements of $\mathfrak{g}$. Both $\mathfrak{g}_{\rm reg}$ and $\mathfrak{g}_{\rm reg,ss}$ are G-invariant dense open subsets of $\mathfrak{g}$. Setting $\mathfrak{h}_{\rm reg} := \mathfrak{h}\cap\mathfrak{g}_{\rm reg}$, $\mathfrak{b}_{\rm reg} := \mathfrak{b}\cap\mathfrak{g}_{\rm reg}$, $\mathfrak{g}_{\rm reg,ss} = G(\mathfrak{h}_{\rm reg})$ and $\mathfrak{g}_{\rm reg} = G(\mathfrak{b}_{\rm reg})$.

• Let $p_1,\dots,p_\ell$ be some homogeneous polynomials generating the algebra ${\rm S}(\mathfrak{g})^G$ of invariant polynomials under G. For $i=1,\dots,\ell$ and for x in $\mathfrak{g}$, denote by $\varepsilon_i(x)$ the element of $\mathfrak{g}$ given by

$$\langle\varepsilon_i(x)\,,y\rangle = \frac{{\rm d}}{{\rm d}t}p_i(x+ty)\,|_{t=0}$$

for all y in $\mathfrak{g}$. Thereby, ε_i is an invariant element of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ under the canonical action of G. According to [Ko63, Theorem 9], for x in $\mathfrak{g}$, x is in $\mathfrak{g}_{\mathrm{reg}}$ if and only if $\varepsilon_1(x), \ldots, \varepsilon_\ell(x)$ are linearly independent. In this case, $\varepsilon_1(x), \ldots, \varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$.

2. On the varieties $\mathcal{X}$ and $G \times_B \mathfrak{b}$

Denote by $\pi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}//G$ and $\pi_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}/W(\mathcal{R})$ the quotient maps, i.e the morphisms defined by the invariants. Recall $\mathfrak{g}//G = \mathfrak{h}/W(\mathcal{R})$, and let $\mathcal{X}$ be the following fiber product:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\overline{\chi}"] \arrow[d, "\overline{\rho}"'] & \mathfrak{g} \arrow[d, "\pi_{\mathfrak{g}}"] \\ \mathfrak{h} \arrow[r, "\pi_{\mathfrak{h}}"'] & \mathfrak{h}/W(\mathcal{R}) \end{tikzcd}$$

where $\overline{\chi}$ and $\overline{\rho}$ are the restriction maps. The actions of G and $W(\mathcal{R})$ on $\mathfrak{g}$ and $\mathfrak{h}$ respectively induce an action of $G \times W(\mathcal{R})$ on $\mathcal{X}$. According to [CZ14, Lemma 2.4], $\mathcal{X}$ is irreducible and normal. Moreover, $\mathcal{X}_{\mathrm{reg}} := \mathfrak{g}_{\mathrm{reg}} \times \mathfrak{h} \cap \mathcal{X}$ is a smooth open subset of $\mathcal{X}$, $\Bbbk[\mathcal{X}]$ is the space of global sections $\mathcal{O}_{G\times_B \mathfrak{b}}$ and $\Bbbk[\mathcal{X}]^G = \mathrm{S}(\mathfrak{h})$. According to [CZ14, Lemma 2.4], the map

$$G \times \mathfrak{b} \longrightarrow \mathcal{X}\ , \qquad (g, x) \longmapsto (g(x), \overline{x})$$

defines through the quotient a projective birational morphism

$$G \times_B \mathfrak{b} \xrightarrow{\ \chi_{\mathrm{n}}\ } \mathcal{X}\ .$$

Lemma 2.1. (i) *The set $\mathfrak{b}_{\mathrm{reg}}$ is a big open subset of $\mathfrak{b}$.*

(ii) *The set $G \times_B \mathfrak{b}_{\mathrm{reg}}$ is a big open subset of $G \times_B \mathfrak{b}$.*

(iii) *The restriction of χ_{n} to $G \times_B \mathfrak{b}_{\mathrm{reg}}$ is an isomorphism onto $\mathcal{X}_{\mathrm{reg}}$.*

(iv) *The restriction of $\pi_{\mathfrak{g}}$ to $\mathfrak{g}_{\mathrm{reg}}$ is a smooth morphism.*

Proof. (i) Let Σ be an irreducible component of $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$. Then Σ is a closed cone invariant under B and $\overline{\Sigma} := \Sigma \cap \mathfrak{h}$ is a closed cone of $\mathfrak{h}$. According to Lemma 1.4, Σ is contained in $\overline{\Sigma} + \mathfrak{u}$. Suppose that Σ has codimension 1 in $\mathfrak{b}$. A contradiction is expected. Then $\overline{\Sigma} = \mathfrak{h}$ or $\overline{\Sigma}$ has codimension 1 in $\mathfrak{h}$. The first case is impossible since $\mathfrak{h} \cap \mathfrak{b}_{\mathrm{reg}}$ is not empty. Hence $\Sigma = \overline{\Sigma} + \mathfrak{u}$ since Σ is irreducible of codimension 1 in $\mathfrak{b}$. As a result, $\mathfrak{u}$ is contained in Σ since $\overline{\Sigma}$ is a closed cone, whence the contradiction since $\mathfrak{u} \cap \mathfrak{b}_{\mathrm{reg}}$ is not empty.

(ii) The complement of $G \times_B \mathfrak{b}_{\mathrm{reg}}$ in $G \times_B \mathfrak{b}$ is equal to $G \times_B \mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$. By (i), $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$, is a B-invariant closed subset of $\mathfrak{b}$ of dimension at most $\dim \mathfrak{b} - 2$. Then $G \times_B \mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$ is a closed subset of $G \times_B \mathfrak{b}$ of codimension at least 2, whence the assertion.

(iii) By definition, $\mathcal{X}_{\mathrm{reg}} = \chi_{\mathrm{n}}(G \times_B \mathfrak{b}_{\mathrm{reg}})$. Let (g_1, x_1) and (g_2, x_2) be in $G \times \mathfrak{b}_{\mathrm{reg}}$ such that $(g_1(x_1), \overline{x_1}) = (g_2(x_2), \overline{x_2})$. For some b_1 and b_2 in B,

$$b_1(x_1)_{\mathrm{s}} = \overline{x_1} \quad \text{and} \quad b_2(x_2)_{\mathrm{s}} = \overline{x_2} = \overline{x_1}.$$

Setting:

$$y_1 := b_1(x_1) \quad \text{and} \quad y_2 := b_2(x_2),$$

$y_2 = b_2 g_2^{-1} g_1 b_1^{-1}(y_1)$ is a regular element of $\mathfrak{g}^{\overline{x_1}}$. In particular, $y_{2,\mathrm{n}}$ and $y_{1,\mathrm{n}}$ are regular nilpotent elements of $\mathfrak{g}^{\overline{x_1}}$ and they are in the borel subalgebra $\mathfrak{b} \cap \mathfrak{g}^{\overline{x_1}}$ of $\mathfrak{g}^{\overline{x_1}}$. Hence $b_2 g_2^{-1} g_1 b_1^{-1}$ is in B and so is $g_2^{-1} g_1$. As a result, the restriction of χ_{n} to $G \times_B \mathfrak{b}_{\mathrm{reg}}$ is injective. So, by Zariski's Main Theorem [Mu88, §9], the restriction of χ_{n} to $G \times_B \mathfrak{b}_{\mathrm{reg}}$ is an isomorphism onto $\mathcal{X}_{\mathrm{reg}}$ since $\mathcal{X}_{\mathrm{reg}}$ is a smooth variety.

(iv) Let x be in $\mathfrak{g}_{\rm reg}$. The kernel of the differential of $\pi_{\mathfrak{g}}$ at x is the orthogonal complement of $\mathfrak{g}^x$ so that the differential of $\pi_{\mathfrak{g}}$ at x is surjective whence the assertion by [H77, Ch. III, Proposition 10.4]. □

Proposition 2.2. (i) *There exists a regular form of top degree, without zero on $\mathcal{X}_{\rm reg}$.*

(ii) *There exists a regular form of top degree, without zero on $G\times_B \mathfrak{b}$.*

Proof. (i) Let ω be a volume form on $\mathfrak{g}$. According to Lemma 2.1,(iv), the restriction of ω to $\mathfrak{g}_{\rm reg}$ is divisible by $\mathrm{d}p_1\wedge\cdots\wedge\mathrm{d}p_\ell$ so that

$$\omega = \alpha\wedge \mathrm{d}p_1\wedge\cdots\wedge\mathrm{d}p_\ell$$

with α a regular relative differential form of top degree with respect to $\pi_{\mathfrak{g}}$. Denoting by $v_1,\ldots,v_\ell$ a basis of $\mathfrak{h}$,

$$\omega' := \alpha\wedge \mathrm{d}v_1\wedge\cdots\wedge\mathrm{d}v_\ell$$

is a regular form of top degree on $\mathcal{X}_{\rm reg}$ since $\mathrm{S}(\mathfrak{g})^G$ identifies with a subalgebra of $\mathrm{S}(\mathfrak{h})$. As $\pi_{\mathfrak{g}}$ and $\overline{\rho}$ have the same fibers and ω has no zero so has ω'.

(ii) By Lemma 2.1,(iii), $\chi_{\mathfrak{n}}^*(\omega')$ is a regular form of top degree on $G\times_B\mathfrak{b}_{\rm reg}$ without zero. Then by Lemma C.1,(ii) and Lemma 2.1,(ii), theres exists a regular form of top degree on $G\times_B\mathfrak{b}$, without zero. □

3. Main varieties and tautological vector bundles

Denote by X the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under B. Since G/B is a projective variety, $G.X$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under G. Set:

$$\mathcal{E}_0 := \{(u,x)\in X\times\mathfrak{b} \mid; x\in u\},\qquad \mathcal{E} := \{(u,x)\in G.X\times\mathfrak{g} \mid; x\in u\}.$$

Then $\mathcal{E}_0$ and $\mathcal{E}$ are the restrictions to X and $G.X$ respectively of the tautological vector bundle of rank ℓ over $\mathrm{Gr}_\ell(\mathfrak{g})$. Denote by π_0 and π the bundle projections:

$$\mathcal{E}_0 \xrightarrow{\ \pi_0\ } X\ ,\qquad \mathcal{E}\xrightarrow{\ \pi\ } G.X\ .$$

Since the map

$$\mathfrak{g}_{\rm reg}\longrightarrow \mathrm{Gr}_\ell(\mathfrak{g})\ ,\qquad x\longmapsto \mathfrak{g}^x$$

is regular, for all x in $\mathfrak{g}_{\rm reg}$, $\mathfrak{g}^x$ is in $G.X$ and for all x in $\mathfrak{b}_{\rm reg}$, $\mathfrak{g}^x$ is in X. Denoting by X' the image of $\mathfrak{b}_{\rm reg}$, $G.X'$ is the image of $\mathfrak{g}_{\rm reg}$ and according to [CZ14, Theorem 1.2], X' and $G.X'$ are smooth big open subsets of X and $G.X$ respectively.

Let τ_0 and τ be the restrictions to $\mathcal{E}_0$ and $\mathcal{E}$ respectively of the canonical projection $\mathrm{Gr}_\ell(\mathfrak{g})\times\mathfrak{g}\to\mathfrak{g}$. Denote by π_* and τ_* the morphisms

$$G\times_B\mathcal{E}_0\xrightarrow{\ \pi_*\ } G\times_B X\ ,\qquad \text{and}\qquad G\times_B\mathcal{E}_0\xrightarrow{\ \tau_*\ }\mathcal{X}$$

defined through the quotients by the maps

$$G\times\mathcal{E}_0\longrightarrow G\times X\ ,\qquad (g,u,x)\longmapsto (g,u),$$

$$G\times\mathcal{E}_0\longrightarrow \mathcal{X}\ ,\qquad (g,u,x)\longmapsto (g(x),\overline{x}).$$

Lemma 3.1. (i) *The morphism τ_0 is a projective and birational morphism from $\mathcal{E}_0$ onto $\mathfrak{b}$.*

(ii) *The morphism τ is a projective and birational morphism from $\mathcal{E}$ onto $\mathfrak{g}$.*

(iii) *The morphism τ_* is a projective and birational morphism from $G\times_B\mathcal{E}_0$ onto $\mathcal{X}$.*

Proof. (i) and (ii) Since X and $G.X$ are projective varieties, τ_0 and τ are projective morphisms. For x in $\mathfrak{g}_{\rm reg}$, $\tau^{-1}(x) = \{\mathfrak{g}^x\}$. Hence τ_0 and τ are birational and their images are $\mathfrak{b}$ and $\mathfrak{g}$ since $\mathfrak{g}_{\rm reg}$ is an open subset of $\mathfrak{g}$.

(iii) The morphism

$$G \times \mathcal{E}_0 \longrightarrow G \times \mathfrak{b} , \qquad (g, u, x) \longmapsto (g, x)$$

defines through the quotient a morphism

$$G \times_B \mathcal{E}_0 \xrightarrow{\tau_1} G \times_B \mathfrak{b} .$$

The varieties $G \times_B \mathcal{E}_0$ and $G \times_B \mathfrak{b}$ are embedded into $G/B \times \mathcal{E}$ and $G/B \times \mathfrak{g}$ respectively as closed subsets and τ_1 is the restriction to $G \times_B \mathcal{E}_0$ of $\mathrm{id}_{G/B}{\times}\tau$. Hence τ_1 is a projective morphism by (ii). As τ_* is the composition of τ_1 and $\chi_{\rm n}$, τ_* is a projective morphism since so is $\chi_{\rm n}$. The map

$$G \times \mathfrak{b}_{\rm reg} \longrightarrow G \times \mathcal{E}_0 , \qquad (g, x) \longmapsto (g, \mathfrak{g}^x, x)$$

defines through the quotient a morphism

$$G \times_B \mathfrak{b}_{\rm reg} \xrightarrow{\mu} G \times_B \mathcal{E}_0 .$$

According to Lemma 2.1,(iii), the restriction of τ_* to $\tau_*^{-1}(\mathcal{X}_{\rm reg})$ is an isomorphism onto $\mathcal{X}_{\rm reg}$ whose inverse is $\mu{\circ}\chi_{\rm n}^{-1}$. In particular, τ_* is birational. □

Denote by $(G.X)_{\rm n}$ the normalization of $G.X$. Let $\mathcal{E}_{\rm n}$ be the following fiber product:

$$\begin{array}{ccc} \mathcal{E}_{\rm n} & \xrightarrow{\nu_{\rm n}} & \mathcal{E} \\ {\scriptstyle \pi_{\rm n}}\downarrow & & \downarrow{\scriptstyle \pi} \\ (G.X)_{\rm n} & \xrightarrow[\nu]{} & G.X \end{array}$$

with ν the normalization morphism, $\nu_{\rm n}$, $\pi_{\rm n}$ the restriction maps.

Proposition 3.2. (i) *The varieties $\mathcal{E}_0$ and X are Gorenstein with rational singularities.*

(ii) *The varieties $\mathcal{E}_{\rm n}$ and $X_{\rm n}$ are Gorenstein with rational singularities.*

(iii) *The varieties $G \times_B \mathcal{E}_0$ and $G \times_B X$ are Gorenstein with rational singularities.*

Proof. According to [C15, Theorem 1.1], X is Gorenstein with rational singularities, then by Lemma D.1,(i) and (iv), so is $\mathcal{E}_0$ as a vector bundle over X. Furthermore, by Lemma D.1,(i) and (iii), $G \times_B X$ is Gorenstein with rational singularities as a fiber bundle over a smooth variety whose fibers are Gorenstein with rational singularities. As a result, by Lemma D.1,(i) and (iv), $G \times_B \mathcal{E}_0$ is Gorenstein with rational singularities as a vector bundle over $G \times_B X$.

Proposition 3.2,(ii) will be proved in Section 5 (see Corollary 5.2). □

4. On the generalized isospectral commuting variety

Let $k \geqslant 2$ be an integer. The variety $G^k \times_{B^k} \mathfrak{b}^k$ identifies with $(G \times_B \mathfrak{b})^k$. Denote by $\chi_{\rm n}^{(k)}$ the morphism

$$G^k \times_{B^k} \mathfrak{b}^k \xrightarrow{\chi_{\rm n}^{(k)}} \mathfrak{X}^{(k)} , \qquad (x_1, \ldots, x_k) \longmapsto (\chi_{\rm n}(x_1), \ldots, \chi_{\rm n}(x_k)).$$

The varitey G/B identifies with the diagonal Δ of $(G/B)^k$ so that $G \times_B \mathfrak{b}^k$ identifies with the restriction to Δ of the vector bundle $G^k \times_{B^k} \mathfrak{b}^{(k)}$ over G/B. Denote by $\gamma_{\rm x}$ the restriction of $\chi_{\rm n}^{(k)}$ to $G \times_B \mathfrak{b}^k$ and by $\mathcal{B}_{\rm x}^{(k)}$ its

image, whence a commutative diagram

$$\begin{tikzcd} G\times_B \mathfrak{b}^k \arrow[rr, "\gamma_{\rm x}"] \arrow[dr, "\gamma"'] & & \mathcal{B}_{\rm x}^{(k)} \arrow[dl, "\eta"] \\ & \mathcal{B}^{(k)} & \end{tikzcd}$$

with η the restriction to $\mathcal{B}_{\rm x}^{(k)}$ of the canonical projection $\mathcal{X}^k \longrightarrow \mathfrak{g}^k$. Let $\iota_{{\rm x},k}$ be the map given by

$$\mathfrak{b}^k \xrightarrow{\iota_{{\rm x},k}} \mathcal{X}^k, \qquad (x_1,\ldots,x_k)\longmapsto (x_1,\ldots,x_k,\overline{x_1},\ldots,\overline{x_k}).$$

According to [CZ14, Lemma 2.7,(i) and Corollary 2.8,(i)], $\iota_{{\rm x},k}$ is a closed embedding of $\mathfrak{b}^k$ into $\mathcal{B}_{\rm x}^{(k)}$ and $\gamma_{\rm x}$ is a projective birational morphism so that $\mathcal{B}_{\rm n}^{(k)}$ is the normalization of $\mathcal{B}_{\rm x}^{(k)}$. Denote by $\mathcal{C}^{(k)}$ the closure of $G.\mathfrak{h}^k$ in $\mathfrak{g}^k$ with respect to the diagonal action of G in $\mathfrak{g}^k$ and set $\mathcal{C}_{\rm x}^{(k)} := \eta^{-1}(\mathcal{C}^{(k)})$. The varieties $\mathcal{C}^{(k)}$ and $\mathcal{C}_{\rm x}^{(k)}$ are called *generalized commuting variety* and *generalized isospectral commuting variety* respectively. For $k=2$, $\mathcal{C}_{\rm x}^{(k)}$ is the isospectral commuting variety considered by M. Haiman in [Ha99, §8] and [Ha02, §7.2]. According to [CZ14, Proposition 5.6], $\mathcal{C}_{\rm x}^{(k)}$ is irreducible and equal to the closure of $G.\iota_{{\rm x},k}(\mathfrak{h}^k)$ in $\mathcal{B}_{\rm x}^{(k)}$.

4.1. We consider the diagonal action of B in $\mathfrak{b}^k$. Let $\mathfrak{X}_{0,k}$ be the closure of $B.\mathfrak{h}^k$ in $\mathfrak{b}^k$. Set:

$$\mathcal{E}^{(k)} := \{(u,x_1,\ldots,x_k)\in G.X\times\mathfrak{g}^k \mid x_1\in u,\ldots,x_k\in u\} \quad\text{and}\quad \mathcal{E}_0^{(k)} := \mathcal{E}^{(k)}\cap X\times\mathfrak{b}^k.$$

Then $\mathcal{E}_0^{(k)}$ and $\mathcal{E}^{(k)}$ are vector bundles over X and $G.X$ respectively. Denote by $\pi_{0,k}$ and π_k respectively their bundle projections. Let $\tau_{0,k}$ and τ_k be the restrictions to $\mathcal{E}_0^{(k)}$ and $\mathcal{E}^{(k)}$ respectively of the canonical projection $\mathrm{Gr}_\ell(\mathfrak{g})\times\mathfrak{g}^k\rightarrow\mathfrak{g}^k$. Denote by $\pi_{*,k}$ and $\tau_{*,k}$ the morphisms

$$G\times_B\mathcal{E}_0^{(k)} \xrightarrow{\pi_{*,k}} G\times_B X, \qquad\text{and}\qquad G\times_B\mathcal{E}_0^{(k)} \xrightarrow{\tau_{*,k}} \mathcal{X}^k$$

defined through the quotients by the maps

$$G\times\mathcal{E}_0^{(k)} \longrightarrow G\times X, \qquad (g,u,x)\longmapsto (g,u),$$

$$G\times\mathcal{E}_0^{(k)} \longrightarrow \mathcal{X}^k, \qquad (g,u,x_1,\ldots,x_k)\longmapsto (g(x_1),\ldots,g(x_k),\overline{x_1},\ldots,\overline{x_k}).$$

Lemma 4.1. (i) *The morphism $\tau_{0,k}$ is a projective morphism from $\mathcal{E}_0^{(k)}$ onto $\mathfrak{X}_{0,k}$.*

(ii) *The morphism τ_k is a projective morphism from $\mathcal{E}^{(k)}$ onto $\mathcal{C}^{(k)}$.*

(iii) *The morphism $\tau_{*,k}$ is a projective morphism from $G\times_B\mathcal{E}_0^{(k)}$ onto $\mathcal{C}_{\rm x}^{(k)}$.*

Proof. (i) Since X is a projective variety, $\tau_{0,k}$ is a projective morphism. Then its image is an irreducible closed subset of $\mathfrak{b}^k$ since $\mathcal{E}_0^{(k)}$ is irreducible as a vector bundle over an irreducible variety. Moreover, $B.\mathfrak{h}^k$ is contained in $\tau_{0,k}(\mathcal{E}_0^{(k)})$ since $\tau_{0,k}(\mathcal{E}_0^{(k)})$ is invariant under B and contains $\mathfrak{h}^k$. As a vector bundle of rank $k\ell$ over X, $\mathcal{E}_0^{(k)}$ has dimension $k\ell+\dim\mathfrak{u}$. Since the restriction to $U\times\mathfrak{h}_{\rm reg}^k$ of the map

$$B\times\mathfrak{h}^k \longrightarrow \mathfrak{b}^k, \qquad (g,x_1,\ldots,x_k)\longmapsto (g(x_1),\ldots,g(x_k))$$

is injective, $\mathfrak{X}_{0,k}$ has dimension $\dim\mathfrak{u}+k\ell$. Hence $\mathfrak{X}_{0,k}$ is the image of $\mathcal{E}_0^{(k)}$ by $\tau_{0,k}$.

(ii) Since $G.X$ is a projective variety, τ_k is a projective morphism. Then its image is an irreducible closed subset of $\mathfrak{g}^k$ since $\mathcal{E}^{(k)}$ is irreducible as a vector bundle over an irreducible variety. Moreover, $G.\mathfrak{h}^k$ is

contained in $\tau_k(\mathcal{E}^{(k)})$ since $\tau_k(\mathcal{E}^{(k)})$ is invariant under G and contains $\mathfrak{h}^k$. As a vector bundle of rank $k\ell$ over $G.X$, $\mathcal{E}^{(k)}$ has dimension $k\ell + 2\dim\mathfrak{u}$. Since the fibers of the restriction to $G\times\mathfrak{h}^k_{\rm reg}$ of the map

$$G\times\mathfrak{h}^k \longrightarrow \mathfrak{g}^k\ , \qquad (g,x_1,\ldots,x_k)\longmapsto (g(x_1),\ldots,g(x_k))$$

have dimension ℓ, $\mathcal{C}^{(k)}$ has dimension $2\dim\mathfrak{u} + k\ell$. Hence $\mathcal{C}^{(k)}$ is the image of $\mathcal{E}^{(k)}$ by τ_k.

(iii) The morphism

$$G\times\mathcal{E}_0^{(k)} \longrightarrow G\times\mathfrak{b}^k\ , \qquad (g,u,x)\longmapsto (g,x)$$

defines through the quotient a morphism

$$G\times_B\mathcal{E}_0^{(k)} \xrightarrow{\ \tau_{1,k}\ } G\times_B\mathfrak{b}^k\ .$$

The varieties $G\times_B\mathcal{E}_0^{(k)}$ and $G\times_B\mathfrak{b}^k$ are embedded into $G/B\times\mathcal{E}^{(k)}$ and $G/B\times\mathfrak{g}^k$ respectively as closed subsets and $\tau_{1,k}$ is the restriction to $G\times_B\mathcal{E}_0^{(k)}$ of ${\rm id}_{G/B}\times\tau_k$. Hence $\tau_{1,k}$ is a projective morphism by (ii). As $\tau_{*,k}$ is the composition of $\tau_{1,k}$ and $\gamma_{\rm x}$, $\tau_{*,k}$ is a projective morphism since so is $\gamma_{\rm x}$. Moreover, by (ii), the image of $\eta\circ\tau_{*,k}$ is equal to $\mathcal{C}^{(k)}$. Hence $\mathcal{C}_{\rm x}^{(k)}$ is the image of $\tau_{*,k}$ since it is irreducible and equal to $\eta^{-1}(\mathcal{C}^{(k)})$. □

4.2. For $j=1,\ldots,k$, denote by $V_{0,j}^{(k)}$ the subset of elements of $\mathfrak{X}_{0,k}$ whose j-th component is in $\mathfrak{b}_{\rm reg}$ and by $V_j^{(k)}$ the subset of elements of $\mathcal{C}^{(k)}$ whose j-th component is in $\mathfrak{g}_{\rm reg}$. Let $W_j^{(k)}$ be the inverse image of $V_j^{(k)}$ by η.

Let σ_j be the automorphism of $\mathfrak{g}^k$ permuting the first and the j-th components of its elements. Then σ_j is equivariant under the diagonal action of G in $\mathfrak{g}^k$ and $\mathfrak{b}^k$ and $\mathfrak{h}^k$ are invariant under σ_j. As a result, $\mathfrak{X}_{0,k}$ is invariant under σ_j and $\sigma_j(V_{0,1}^{(k)}) = V_{0,j}^{(k)}$. In the same way, $\mathcal{C}^{(k)}$ is invariant under σ_j and $\sigma_j(V_1^{(k)}) = V_j^{(k)}$. The map

$$G\times\mathfrak{b}^k \longrightarrow G\times\mathfrak{b}^k\ , \qquad (g,x)\longmapsto (g,\sigma_j(x))$$

defines through the quotient an automorphism of $G\times_B\mathfrak{b}^k$. Denote again by σ_j this automorphism and the restriction to $\mathcal{X}^k$ of the automorphism $(x,y)\mapsto(\sigma_j(x),\sigma_j(y))$ of $\mathfrak{g}^k\times\mathfrak{h}^k$. Since $\mathcal{B}_{\rm x}^{(k)}$ is contained in $\mathcal{X}^k$ and $\gamma_{\rm x}$ is a morphism from $G\times_B\mathfrak{b}^k$ to $\mathcal{X}^k$ such that $\gamma_{\rm x}\circ\sigma_j = \sigma_j\circ\gamma_{\rm x}$, $\mathcal{B}_{\rm x}^{(k)}$ is invariant under σ_j. In the same way, $\sigma_j\circ\gamma = \gamma\circ\sigma_j$ and $\mathcal{B}^{(k)}$ is invariant under σ_j. As a result $\sigma_j\circ\eta = \eta\circ\sigma_j$, $\mathcal{C}_{\rm x}^{(k)}$ is invariant under σ_j and $\sigma_j(W_1^{(k)}) = W_j^{(k)}$.

Lemma 4.2. *Let $j=1,\ldots,k$.*

(i) *The set $V_{0,j}^{(k)}$ is a smooth open subset of $\mathfrak{X}_{0,k}$. Moreover there exists a regular differential form of top degree on $V_{0,j}^{(k)}$, without zero.*

(ii) *The set $V_j^{(k)}$ is a smooth open subset of $\mathcal{C}^{(k)}$. Moreover there exists a regular differential form of top degree on $V_j^{(k)}$, without zero.*

(iii) *The set $W_j^{(k)}$ is a smooth open subset of $\mathcal{C}_{\rm x}^{(k)}$. Moreover there exists a regular differential form of top degree on $W_j^{(k)}$, without zero.*

Proof. According to the above remarks, we can suppose $j=1$.

(i) By definition, $V_{0,1}^{(k)}$ is the intersection of $\mathfrak{X}_{0,k}$ and the open subset $\mathfrak{b}_{\rm reg}\times\mathfrak{b}^{k-1}$ of $\mathfrak{b}^k$. Hence $V_{0,1}^{(k)}$ is an open subset of $\mathfrak{X}_{0,k}$. For x_1 in $\mathfrak{b}_{\rm reg}$, $(x_1,\ldots,x_k)$ is in $V_{0,1}^{(k)}$ if and only if $x_2,\ldots,x_k$ are in $\mathfrak{g}^{x_1}$ by Lemma 4.1,(i)

since $\mathfrak{g}^{x_1}$ is in X. According to [Ko63, Theorem 9], for x in $\mathfrak{b}_{\rm reg}$, $\varepsilon_1(x),\dots,\varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$ and $\mathfrak{g}^x$ is contained in $\mathfrak{b}$. Hence the map

$$\begin{array}{ccc} \mathfrak{b}_{\rm reg}\times \mathrm{M}_{k-1,\ell}(\Bbbk) & \xrightarrow{\theta_0} & V_{0,1}^{(k)} ,\\ (x,(a_{i,j},1\leqslant i\leqslant k-1,1\leqslant j\leqslant \ell)) & \longmapsto & (x,\sum_{j=1}^{\ell} a_{1,j}\varepsilon_j(x),\dots,\sum_{j=1}^{\ell} a_{k-1,j}\varepsilon_j(x)) \end{array}$$

is a bijective morphism. The open subset $\mathfrak{b}_{\rm reg}$ has a cover by open subsets V such that for some $e_1,\dots,e_n$ in $\mathfrak{b}$, $\varepsilon_1(x),\dots,\varepsilon_\ell(x),e_1,\dots,e_n$ is a basis of $\mathfrak{b}$ for all x in V. Then there exist regular functions $\varphi_1,\dots,\varphi_\ell$ on $V\times\mathfrak{b}$ such that

$$v-\sum_{j=1}^{\ell}\varphi_j(x,v)\varepsilon_j(x)\in \operatorname{span}(e_1,\dots,e_n)$$

for all (x,v) in $V\times\mathfrak{b}$, so that the restriction of θ_0 to $V\times \mathrm{M}_{k-1,\ell}(\Bbbk)$ is an isomorphism onto $\mathfrak{X}_{0,k}\cap V\times\mathfrak{b}^{k-1}$ whose inverse is

$$(x_1,\dots,x_k)\longmapsto (x_1,((\varphi_1(x_1,x_i),\dots,\varphi_\ell(x_1,x_i)),i=2,\dots,k)).$$

As a result, θ_0 is an isomorphism and $V_{0,1}^{(k)}$ is a smooth variety. Since $\mathfrak{b}_{\rm reg}$ is a smooth open subset of the vector space $\mathfrak{b}$, there exists a regular differential form ω of top degree on $\mathfrak{b}_{\rm reg}\times\mathrm{M}_{k-1,\ell}(\Bbbk)$, without zero. Then $\theta_{0*}(\omega)$ is a regular differential form of top degree on $V_{0,1}^{(k)}$, without zero.

(ii) By definition, $V_1^{(k)}$ is the intersection of $\mathcal{C}^{(k)}$ and the open subset $\mathfrak{g}_{\rm reg}\times\mathfrak{g}^{k-1}$ of $\mathfrak{g}^k$. Hence $V_1^{(k)}$ is an open subset of $\mathcal{C}^{(k)}$. For x_1 in $\mathfrak{g}_{\rm reg}$, $(x_1,\dots,x_k)$ is in $V_1^{(k)}$ if and only if $x_2,\dots,x_k$ are in $\mathfrak{g}^{x_1}$ by Lemma 4.1,(ii) since $\mathfrak{g}^{x_1}$ is in $G.X$. According to [Ko63, Theorem 9], for x in $\mathfrak{g}_{\rm reg}$, $\varepsilon_1(x),\dots,\varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$. Hence the map

$$\begin{array}{ccc} \mathfrak{g}_{\rm reg}\times \mathrm{M}_{k-1,\ell}(\Bbbk) & \xrightarrow{\theta} & V_{1}^{(k)} ,\\ (x,(a_{i,j},1\leqslant i\leqslant k-1,1\leqslant j\leqslant \ell)) & \longmapsto & (x,\sum_{j=1}^{\ell} a_{1,j}\varepsilon_j(x),\dots,\sum_{j=1}^{\ell} a_{k-1,j}\varepsilon_j(x)) \end{array}$$

is a bijective morphism. The open subset $\mathfrak{g}_{\rm reg}$ has a cover by open subsets V such that for some $e_1,\dots,e_{2n}$ in $\mathfrak{g}$, $\varepsilon_1(x),\dots,\varepsilon_\ell(x),e_1,\dots,e_{2n}$ is a basis of $\mathfrak{g}$ for all x in V. Then there exist regular functions $\varphi_1,\dots,\varphi_\ell$ on $V\times\mathfrak{g}$ such that

$$v-\sum_{j=1}^{\ell}\varphi_j(x,v)\varepsilon_j(x)\in \operatorname{span}(e_1,\dots,e_{2n})$$

for all (x,v) in $V\times\mathfrak{g}$, so that the restriction of θ to $V\times \mathrm{M}_{k-1,\ell}(\Bbbk)$ is an isomorphism onto $\mathcal{C}^{(k)}\cap V\times\mathfrak{b}^{k-1}$ whose inverse is

$$(x_1,\dots,x_k)\longmapsto (x_1,((\varphi_1(x_1,x_i),\dots,\varphi_\ell(x_1,x_i)),i=2,\dots,k)).$$

As a result, θ is an isomorphism and $V_1^{(k)}$ is a smooth variety. Since $\mathfrak{g}_{\rm reg}$ is a smooth open subset of the vector space $\mathfrak{g}$, there exists a regular differential form ω of top degree on $\mathfrak{g}_{\rm reg}\times\mathrm{M}_{k-1,\ell}(\Bbbk)$, without zero. Then $\theta_*(\omega)$ is a regular differential form of top degree on $V_1^{(k)}$, without zero.

(iii) Since $\mathcal{X}_{\rm reg}$ is the inverse image of $\mathfrak{g}_{\rm reg}$ by the canonical projection $\mathcal{X}\longrightarrow\mathfrak{g}$, $W_1^{(k)}$ is the intersection of $\mathcal{C}_{\rm x}^{(k)}$ and $\mathcal{X}_{\rm reg}\times\mathcal{X}^{k-1}$. Hence $W_1^{(k)}$ is an open subset of $\mathcal{C}_{\rm x}^{(k)}$. Moreover, $W_1^{(k)}$ is the image of $G\times_B V_{0,1}^{(k)}$ by $\gamma_{\rm x}$. Since the maps $\varepsilon_1,\dots,\varepsilon_\ell$ are G-equivariant, the map

$$\begin{array}{ccc} G\times\mathfrak{b}_{\rm reg}\times \mathrm{M}_{k-1,\ell}(\Bbbk) & \longrightarrow & W_1^{(k)} ,\\ (g,x,a_{i,j},1\leqslant i\leqslant k-1,1\leqslant j\leqslant\ell) & \longmapsto & \gamma_{\rm x}(\overline{g,\theta_0(x,a_{i,j},1\leqslant i\leqslant k-1,1\leqslant j\leqslant\ell)}) \end{array}$$

defines through the quotient a surjective morphism

$$G\times_B \mathfrak{b}_{\rm reg}\times \mathrm{M}_{k-1,\ell}(\Bbbk) \xrightarrow{\theta_{\rm x}} W_1^{(k)} \ .$$

Let ϖ_2 be the canonical projection

$$\mathfrak{g}_{\rm reg}\times \mathrm{M}_{k-1,\ell}(\Bbbk) \longrightarrow \mathrm{M}_{k-1,\ell}(\Bbbk) \ .$$

According to Lemma 2.1,(iii), the restriction of $\chi_{\rm n}$ to $G\times_B \mathfrak{b}_{\rm reg}$ is an isomorphism onto $\mathcal{X}_{\rm reg}$. So denote by ϖ_1 the morphism

$$W_1^{(k)} \xrightarrow{\varpi_1} G\times_B \mathfrak{b}_{\rm reg} \ , \qquad (x_1,\dots,x_k,y_1,\dots,y_k)\longmapsto \chi_{\rm n}^{-1}(x_1,y_1).$$

Then $\theta_{\rm x}$ is an isomorphism whose inverse is given by

$$x\longmapsto (\varpi_1(x), \varpi_2\circ\theta^{-1}\circ\eta(x))$$

since $\eta(W_1^{(k)}) = V_1^{(k)}$. In particular, $W_1^{(k)}$ is a smooth open subset of $\mathcal{C}_{\rm x}^{(k)}$. According to Proposition 2.2,(ii), there exists a regular differential form ω of top degree on $G\times_B \mathfrak{b}_{\rm reg}\times M_{k-1,\ell}(k)$, without zero. Then $\theta_{{\rm x}*}(\omega)$ is a regular differential form of top degree on $W_1^{(k)}$, without zero. □

Corollary 4.3. *Let $j = 1,\dots,k$.*

(i) *The morphism $\tau_{0,k}$ is birational. More precisely, the restriction of $\tau_{0,k}$ to $\tau_{0,k}^{-1}(V_{0,j}^{(k)})$ is an isomorphism onto $V_{0,j}^{(k)}$.*

(ii) *The morphism τ_k is birational. More precisely, the restriction of τ_k to $\tau_k^{-1}(V_j^{(k)})$ is an isomorphism onto $V_j^{(k)}$.*

(iii) *The morphism $\tau_{*,k}$ is birational. More precisely, the restriction of $\tau_{*,k}$ to $\tau_{*,k}^{-1}(W_j^{(k)})$ is an isomorphism onto $W_j^{(k)}$.*

Proof. According to the above remarks, we can suppose $j = 1$.

(i) For $(x_1,\dots,x_k)$ in $V_{0,1}^{(k)}$ and for u in X containing $x_1,\dots,x_k$, $u = \mathfrak{g}^{x_1}$ since x_1 is regular. Hence the restriction of $\tau_{0,k}$ to $\tau_{0,k}^{-1}(V_{0,1}^{(k)})$ is injective. According to Lemma 4.1,(i), $V_{0,1}^{(k)}$ is the image of $\tau_{0,k}^{-1}(V_{0,1}^{(k)})$ by $\tau_{0,k}$. So, by Lemma 4.2,(i) and Zariski's Main Theorem [Mu88, §9], the restriction of $\tau_{0,k}$ to $\tau_{0,k}^{-1}(V_{0,1}^{(k)})$ is an isomorphism onto $V_{0,1}^{(k)}$.

(ii) For $(x_1,\dots,x_k)$ in $V_1^{(k)}$ and for u in $G.X$ containing $x_1,\dots,x_k$, $u = \mathfrak{g}^{x_1}$ since x_1 is regular. Hence the restriction of τ_k to $\tau_k^{-1}(V_1^{(k)})$ is injective. According to Lemma 4.1,(ii), $V_1^{(k)}$ is the image of $\tau_k^{-1}(V_1^{(k)})$ by τ_k. So, by Lemma 4.2,(ii) and Zariski's Main Theorem [Mu88, §9], the restriction of τ_k to $\tau_k^{-1}(V_1^{(k)})$ is an isomorphism onto $V_1^{(k)}$.

(iii) The variety $G\times_B\mathcal{E}_0^{(k)}$ identifies to a closed subvariety of $G/B\times\mathcal{E}^{(k)}$. For $(x_1,\dots,x_k,y_1,\dots,y_k)$ in $W_1^{(k)}$ and (u,v) in $G/B\times G.X$ such that $(u,v,x_1,\dots,x_k)$ is in $G\times_B\mathcal{E}_0^{(k)}$, (x_1,y_1) is in $\mathcal{X}_{\rm reg}$, $\chi_{\rm n}(u,x_1) = (x_1,y_1)$ and $v = \mathfrak{g}^{x_1}$ since x_1 is regular. Moreover, u is unique by Lemma 2.1,(iii). Hence the restriction of $\tau_{*,k}$ to $\tau_{*,k}^{-1}(W_1^{(k)})$ is injective. According to Lemma 4.1,(iii), $W_1^{(k)}$ is the image of $\tau_{*,k}^{-1}(W_1^{(k)})$ by $\tau_{*,k}$. So, by Lemma 4.2,(iii) and Zariski's Main Theorem [Mu88, §9], the restriction of $\tau_{*,k}$ to $\tau_{*,k}^{-1}(W_1^{(k)})$ is an isomorphism onto $W_1^{(k)}$. □

Set:

$$V_0^{(k)} := V_{0,1}^{(k)}\cup V_{0,2}^{(k)}, \qquad V^{(k)} := V_1^{(k)}\cup V_2^{(k)}, \qquad W^{(k)} := W_1^{(k)}\cup W_2^{(k)}.$$

Lemma 4.4. (i) *The set* $\tau_{0,k}^{-1}(V_0^{(k)})$ *is a big open subset of* $\mathcal{E}_0^{(k)}$.

(ii) *The set* $\tau_k^{-1}(V^{(k)})$ *is a big open subset of* $\mathcal{E}^{(k)}$.

(i) *The set* $\tau_{*,k}^{-1}(W^{(k)})$ *is a big open subset of* $G\times_B \mathcal{E}_0^{(k)}$.

Proof. (i) Let Σ be an irreducible component of $\mathcal{E}_0^{(k)} \setminus \tau_{0,k}^{-1}(V_0^{(k)})$. Since $V_0^{(k)}$ is a B-invariant open cone, Σ is a B-invariant closed subset of $\mathcal{E}_0^{(k)}$ such that $\Sigma\cap\pi_{0,k}^{-1}(u)$ is a closed cone of $\pi_{0,k}^{-1}(u)$ for all u in $\pi_{0,k}(\Sigma)$. As a result $\pi_{0,k}(\Sigma)$ is a closed subset of X. Indeed, $\pi_{0,k}(\Sigma)\times\{0\} = \Sigma\cap X\times\{0\}$. For u in $\pi_{0,k}(\Sigma)$, denote by Σ_u the closed subvariety of u^k such that $\pi_{0,k}^{-1}(u)\cap\Sigma = \{u\}\times\Sigma_u$.

Suppose that Σ has codimension 1 in $\mathcal{E}_0^{(k)}$. A contradiction is expected. Then $\pi_{0,k}(\Sigma)$ has codimension at most 1 in X. Since X' is a big open subset of X, for all u in a dense open subset of $\pi_{0,k}(\Sigma)$, $u\cap\mathfrak{b}_{\rm reg}$ is not empty. If $\pi_{0,k}(\Sigma)$ has codimension 1 in X, then $\Sigma_u = u^k$ for all u in $\pi_{0,k}(\Sigma)$. Hence $\pi_{0,k}(\Sigma) = X$ and for all u in a dense open subset of X', Σ_u has dimension $k\ell-1$. For such u, the image of Σ_u by the first projection onto u is not dense in u since $u\cap\mathfrak{b}_{\rm reg}$ is not empty. Hence the image of Σ_u by the second projection is equal to u since Σ_u has codimension 1 in u^k. It is impossible since this image is contained in $u\setminus\mathfrak{b}_{\rm reg}$.

(ii) Let Σ be an irreducible component of $\mathcal{E}^{(k)}\setminus\tau_k^{-1}(V^{(k)})$. Since $V^{(k)}$ is a G-invariant open cone, Σ is a G-invariant closed subset of $\mathcal{E}^{(k)}$ such that $\Sigma\cap\pi_k^{-1}(u)$ is a closed cone of $\pi_k^{-1}(u)$ for all u in $\pi_k(\Sigma)$. As a result $\pi_k(\Sigma)$ is a closed subset of $G.X$. Indeed, $\pi_k(\Sigma)\times\{0\} = \Sigma\cap G.X\times\{0\}$. For u in $\pi_k(\Sigma)$, denote by Σ_u the closed subvariety of u^k such that $\pi_k^{-1}(u)\cap\Sigma = \{u\}\times\Sigma_u$.

Suppose that Σ has codimension 1 in $\mathcal{E}^{(k)}$. A contradiction is expected. Then $\pi_k(\Sigma)$ has codimension at most 1 in X. Since $G.X'$ is a big open subset of $G.X$, for all u in a dense open subset of $\pi_k(\Sigma)$, $u\cap\mathfrak{g}_{\rm reg}$ is not empty. If $\pi_k(\Sigma)$ has codimension 1 in $G.X$, then $\Sigma_u = u^k$ for all u in $\pi_k(\Sigma)$. Hence $\pi_k(\Sigma) = G.X$ and for all u in a dense open subset of $G.X'$, Σ_u has dimension $k\ell-1$. For such u, the image of Σ_u by the first projection onto u is not dense in u since $u\cap\mathfrak{g}_{\rm reg}$ is not empty. Hence the image of Σ_u by the second projection is equal to u since Σ_u has codimension 1 in u^k. It is impossible since this image is contained in $u\setminus\mathfrak{g}_{\rm reg}$.

(iii) Let Σ be an irreducible component of $G\times_B\mathcal{E}_0^{(k)}\setminus\tau_{*,k}^{-1}(W^{(k)})$. Since $W^{(k)}$ is a G-invariant open cone, Σ is a G-invariant closed subset of $G\times_B\mathcal{E}_0^{(k)}$. So, for some B-invariant closed subset Σ_0 of $\mathcal{E}_0^{(k)}$, $\Sigma = G\times_B\Sigma_0$. Moreover, Σ_0 is contained in $\mathcal{E}_{0,k}\setminus\tau_{0,k}^{-1}(V_0^{(k)})$. According to (i), Σ_0 has codimension at least 2 in $\mathcal{E}_0^{(k)}$. Hence Σ has codimension at least 2 in $G\times_B\mathcal{E}_0^{(k)}$. □

4.3. For $2\leqslant k'\leqslant k$, the projection

$$\mathfrak{g}^k \longrightarrow \mathfrak{g}^{k'} , \qquad (x_1,\ldots,x_k)\longmapsto(x_1,\ldots,x_{k'})$$

induces the projections

$$\mathfrak{X}_{0,k}\longrightarrow\mathfrak{X}_{0,k'} , \qquad V_{0,j}^{(k)}\longrightarrow V_{0,j}^{(k')} .$$

Set:

$$V_{0,1,2}^{(k)} := V_{0,1}^{(k)}\cap V_{0,2}^{(k)}.$$

Lemma 4.5. *Let* ω *be a regular differential form of top degree on* $V_{0,1}^{(k)}$*, without zero. Denote by* ω' *its restriction to* $V_{0,1,2}^{(k)}$.

(i) *For* φ *in* $\Bbbk[V_{0,1}^{(k)}]$*, if* φ *has no zero then* φ *is in* $\Bbbk^*$.

(ii) *For some invertible element* ψ *of* $\Bbbk[V_{0,1,2}^{(2)}]$, $\omega' = \psi\sigma_{2*}(\omega')$.

(iii) *The function* $\psi(\psi\circ\sigma_2)$ *on* $V_{0,1,2}^{(k)}$ *is equal to* 1.

Proof. The existence of ω results from Lemma 4.2,(i).

(i) According to Lemma 4.2,(i), there is an isomorphism θ_0 from $\mathfrak{b}_{\rm reg}\times \mathrm{M}_{k-1,\ell}(\Bbbk)$ onto $V_{0,1}^{(k)}$. Since φ is invertible, $\varphi\circ\theta_0$ is an invertible element of $\Bbbk[\mathfrak{b}_{\rm reg}]$. According to Lemma 2.1,(i), $\Bbbk[\mathfrak{b}_{\rm reg}]=\Bbbk[\mathfrak{b}]$. Hence φ is in $\Bbbk^*$.

(ii) The open subset $V_{0,1,2}^{(k)}$ is invariant under σ_2 so that ω' and $\sigma_{2*}(\omega')$ are regular differential forms of top degree on $V_{0,1,2}^{(k)}$, without zero. Then for some invertible element ψ of $\Bbbk[V_{0,1,2}^{(k)}]$, $\omega'=\psi\sigma_{2*}(\omega')$. Let O_2 be the set of elements $(x,a_{i,j},1\leqslant i\leqslant k-1,1\leqslant j\leqslant \ell)$ of $\mathfrak{b}_{\rm reg}\times\mathrm{M}_{k-1,\ell}(\Bbbk)$ such that

$$a_{1,1}\varepsilon_1(x)+\cdots+a_{1,\ell}\varepsilon_\ell(x)\in\mathfrak{b}_{\rm reg}.$$

Then O_2 is the inverse image of $V_{0,1,2}^{(k)}$ by θ_0. As a result, $\Bbbk[V_{0,1,2}^{(k)}]$ is a polynomial algebra over $\Bbbk[V_{0,1,2}^{(2)}]$ since for $k=2$, O_2 is the inverse image by θ_0 of $V_{0,1,2}^{(2)}$. Hence ψ is in $\Bbbk[V_{0,1,2}^{(2)}]$ since ψ is invertible.

(iii) Since the restriction of σ_2 to $V_{0,1,2}^{(k)}$ is an involution,

$$\sigma_{2*}(\omega')=(\psi\circ\sigma_2)\omega'=(\psi\circ\sigma_2)\psi\sigma_{2*}(\omega'),$$

whence $(\psi\circ\sigma_2)\psi=1$. □

Corollary 4.6. *The function ψ is invariant under the action of B in $V_{0,1,2}^{(k)}$ and for some sequence $m_\alpha,\alpha\in\mathcal{R}_+$ in $\mathbb{Z}$,*

$$\psi(x_1,\ldots,x_k)=\pm\prod_{\alpha\in\mathcal{R}_+}(\alpha(x_1)\alpha(x_2)^{-1})^{m_\alpha},$$

for all $(x_1,\ldots,x_k)$ in $\mathfrak{h}_{\rm reg}^2\times\mathfrak{h}^{k-2}$.

Proof. First of all, since $V_{0,1}^{(k)}$ and $V_{0,2}^{(k)}$ are invariant under the action of B in $\mathfrak{X}_{0,k}$, so is $V_{0,1,2}^{(k)}$. Let g be in B. Since ω has no zero, $g.\omega=p_g\omega$ for some invertible element p_g of $\Bbbk[V_{0,1}^{(k)}]$. By Lemma 4.5,(i), p_g is in $\Bbbk^*$. Since σ_2 is a B-equivariant isomorphism from $V_{0,1}^{(k)}$ onto $V_{0,2}^{(k)}$,

$$g.\sigma_{2*}(\omega)=p_g\sigma_{2*}(\omega)\quad\text{and}\quad p_g\omega'=g.\omega'=(g.\psi)g.\sigma_{2*}(\omega')=p_g(g.\psi)\sigma_{2*}(\omega'),$$

whence $g.\psi=\psi$.

The open subset $\mathfrak{h}_{\rm reg}^2$ of $\mathfrak{h}^2$ is the complement of the nullvariety of the function

$$(x,y)\longmapsto\prod_{\alpha\in\mathcal{R}_+}\alpha(x)\alpha(y).$$

Then, by Lemma 4.5,(ii), for some a in $\Bbbk^*$ and for some sequences $m_\alpha,\alpha\in\mathcal{R}_+$ and $n_\alpha,\alpha\in\mathcal{R}_+$ in $\mathbb{Z}$,

$$\psi(x_1,\ldots,x_k)=a\prod_{\alpha\in\mathcal{R}_+}\alpha(x_1)^{m_\alpha}\alpha(x_2)^{n_\alpha},$$

for all $(x_1,\ldots,x_k)$ in $\mathfrak{h}_{\rm reg}^2\times\mathfrak{h}^{k-2}$. Then, by Lemma 4.5,(iii),

$$a^2\prod_{\alpha\in\mathcal{R}_+}\alpha(x)^{m_\alpha+n_\alpha}\alpha(y)^{m_\alpha+n_\alpha}=1,$$

for all (x,y) in $\mathfrak{h}_{\rm reg}^2$. Hence $a^2=1$ and $m_\alpha+n_\alpha=0$ for all α in $\mathcal{R}_+$. □

For α a positive root, denote by $\mathfrak{h}_\alpha$ the kernel of α and set:

$$V_\alpha:=\mathfrak{h}_\alpha\oplus\mathfrak{g}^\alpha.$$

Denote by θ_α the map

$$\Bbbk\xrightarrow{\ \theta_\alpha\ }X\ ,\qquad t\longmapsto\exp(t\operatorname{ad}x_\alpha).\mathfrak{h}.$$

According to [Sh94, Ch. VI, Theorem 1], θ_α has a regular extension to $\mathbb{P}^1(\Bbbk)$. Set $Z_\alpha := \theta_\alpha(\mathbb{P}^1(\Bbbk))$. Denote again by α the element of $\mathfrak{b}^*$ extending α and equal to 0 on $\mathfrak{u}$.

Lemma 4.7. *Let α be in $\mathcal{R}_+$ and let x_0 and y_0 be subregular in $\mathfrak{h}_\alpha$. Set:*

$$E := \Bbbk x_0 \oplus \Bbbk H_\alpha \oplus \mathfrak{g}^\alpha, \quad E_* := x_0 \oplus \Bbbk H_\alpha \oplus \mathfrak{g}^\alpha, \quad E_{*,1} := x_0 \oplus \Bbbk H_\alpha \oplus (\mathfrak{g}^\alpha \setminus \{0\}), \quad E_{*,2} = y_0 \oplus \Bbbk H_\alpha \oplus (\mathfrak{g}^\alpha \setminus \{0\}).$$

(i) *For x in E_*, the centralizer of x in $\mathfrak{b}$ is contained in $\mathfrak{h}_\alpha + E$.*
(ii) *For V subspace of dimension ℓ of $\mathfrak{h}_\alpha + E$, V is in X if and only if it is in Z_α.*
(iii) *The intersection of $E_{*,1} \times E_{*,2}$ and $\mathfrak{X}_{0,2}$ is the nullvariety of the function*

$$(x, y) \longmapsto \langle x_{-\alpha}, y\rangle \alpha(x) - \langle x_{-\alpha}, x\rangle \alpha(y)$$

on $E_{*,1} \times E_{*,2}$.

Proof. (i) If x is regular semisimple, its component on H_α is different from 0 so that $\mathfrak{g}^x = \theta_\alpha(t)$ for some t in $\Bbbk$. Suppose that x is not regular semisimple. Then x is in $x_0 + \mathfrak{g}^\alpha$. Hence $\mathfrak{g}^x \cap \mathfrak{b}$ is contained in $\mathfrak{h}_\alpha + E$ since so is $\mathfrak{g}^{x_0} \cap \mathfrak{b}$.

(ii) All element of Z_α is contained in $\mathfrak{h}_\alpha + E$. Let V be an element of X, contained in $\mathfrak{h}_\alpha + E$. According to [CZ14, Corollary 4.3], V is an algebraic commutative subalgebra of dimension ℓ of $\mathfrak{b}$. By (i), $V = \theta_\alpha(t)$ for some t in $\Bbbk$ if V is a Cartan subalgebra. Otherwise, x_α is in V. Then $V = \theta_\alpha(\infty)$ since $\theta_\alpha(\infty)$ is the centralizer of x_α in $\mathfrak{h}_\alpha + E$.

(iii) Let (x, y) be in $E_{*,1} \times E_{*,2} \cap \mathfrak{X}_{0,2}$. According to Lemma 4.1,(i), for some V in X, x and y are in V. By (i) and (ii), $V = \theta_\alpha(t)$ for some t in $\mathbb{P}^1(\Bbbk)$. For t in $\Bbbk$,

$$x = x_0 + s(H_\alpha - 2t x_\alpha) \quad \text{and} \quad y = y_0 + s'(H_\alpha - 2t x_\alpha)$$

for some s, s' in $\Bbbk$, whence the equality of the assertion. For $t = \infty$,

$$x = x_0 + s x_\alpha \quad \text{and} \quad y = y_0 + s' x_\alpha$$

for some s, s' in $\Bbbk$ so that $\alpha(x) = \alpha(y) = 0$. Conversely, let (x, y) be in $E_{*,1} \times E_{*,2}$ such that

$$\langle x_{-\alpha}, y\rangle \alpha(x) - \langle x_{-\alpha}, x\rangle \alpha(y) = 0.$$

If $\alpha(x) = 0$ then $\alpha(y) = 0$ and x and y are in $V_\alpha = \theta_\alpha(\infty)$. If $\alpha(x) \neq 0$, then $\alpha(y) \neq 0$ and

$$x \in \theta_\alpha(-\frac{\langle x_{-\alpha}, x\rangle}{\alpha(x)}) \quad \text{and} \quad y \in \theta_\alpha(-\frac{\langle x_{-\alpha}, x\rangle}{\alpha(x)}),$$

whence the assertion. □

Proposition 4.8. *There exists on $V_0^{(k)}$ a regular differential form of top degree without zero.*

Proof. According to Corollary 4.6, it suffices to prove $m_\alpha = 0$ for all α in $\mathcal{R}_+$. Indeed, if so, by Corollary 4.6, $\psi = \pm 1$ on the open subset $B.(\mathfrak{h}_{\rm reg}^2 \times \mathfrak{h}^{k-2})$ of $V_0^{(k)}$ so that $\psi = \pm 1$ on $V_{0,1,2}^{(k)}$. Then, by Lemma 4.5,(ii), ω and $\pm\sigma_{2*}(\omega)$ have the same restriction to $V_{0,1,2}^{(k)}$ so that there exists a regular differential form of top degree $\tilde{\omega}$ on $V_0^{(k)}$ whose restrictions to $V_{0,1}^{(k)}$ and $V_{0,2}^{(k)}$ are ω and $\pm\sigma_{2*}(\omega)$ respectively. Moreover, $\tilde{\omega}$ has no zero since so has ω.

Since ψ is in $\Bbbk[V_{0,1,2}^{(2)}]$ by Lemma 4.5,(ii), we can suppose $k = 2$. Let α be in $\mathcal{R}_+$, E, E_*,$E_{*,1}$, $E_{*,2}$ as in Lemma 4.7. Suppose $m_\alpha \neq 0$. A contradiction is expected. According to Lemma 4.7,(iii), the restriction of ψ to $E_{*,1} \times E_{*,2} \cap V_{0,1,2}^{(2)}$ is given by

$$\psi(x, y) = a\langle x_{-\alpha}, x\rangle^m \langle x_{-\alpha}, y\rangle^n,$$

with a in $\Bbbk^*$ and (m,n) in $\mathbb{Z}^2$ since ψ is an invertible element of $\Bbbk[V_{0,1,2}^{(2)}]$. According to Lemma 4.5,(iii), $n=-m$ and $a=\pm 1$. Interchanging the role of x and y, we can suppose m in $\mathbb{N}$. For (x,y) in $E_{*,1}\times E_{*,2}\cap V_{0,1,2}^{(2)}$ such that $\alpha(x)\neq 0$, $\alpha(y)\neq 0$ and

$$\psi(x,y) = \pm\langle x_{-\alpha}\, , x\rangle^m\Big(\frac{\langle x_{-\alpha}\, , x\rangle\alpha(y)}{\alpha(x)}\Big)^{-m} = \pm\alpha(x)^m\alpha(y)^{-m}.$$

As a result, by Corollary 4.6, for x in $x_0+\Bbbk^* H_\alpha$ and y in $y_0+\Bbbk^* H_\alpha$,

$$\pm\,\alpha(x)^m\alpha(y)^{-m} = \pm\prod_{\gamma\in\mathcal{R}_+}\gamma(x)^{m_\gamma}\gamma(y)^{-m_\gamma}. \tag{1}$$

For γ in $\mathcal{R}_+$,

$$\gamma(x) = \gamma(x_0)+\frac{1}{2}\alpha(x)\gamma(H_\alpha) \quad\text{and}\quad \gamma(y) = \gamma(y_0)+\frac{1}{2}\alpha(y)\gamma(H_\alpha).$$

Since m is in $\mathbb{N}$,

$$\begin{aligned}\pm\,\alpha(x)^m\prod_{\substack{\gamma\in\mathcal{R}_+\\ m_\gamma>0}}(\gamma(y_0)+\frac{1}{2}\alpha(y)\gamma(H_\alpha))^{m_\gamma}\prod_{\substack{\gamma\in\mathcal{R}_+\\ m_\gamma<0}}(\gamma(x_0)+\frac{1}{2}\alpha(x)\gamma(H_\alpha))^{-m_\gamma} = \\ \pm\alpha(y)^m\prod_{\substack{\gamma\in\mathcal{R}_+\\ m_\gamma>0}}(\gamma(x_0)+\frac{1}{2}\alpha(x)\gamma(H_\alpha))^{m_\gamma}\prod_{\substack{\gamma\in\mathcal{R}_+\\ m_\gamma<0}}(\gamma(y_0)+\frac{1}{2}\alpha(y)\gamma(H_\alpha))^{-m_\gamma}.\end{aligned} \tag{2}$$

For m_α positive, the terms of lowest degree in $(\alpha(x),\alpha(y))$ of left and right sides are

$$\pm\alpha(x)^m\alpha(y)^{m_\alpha}\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma>0}}\gamma(y_0)^{m_\gamma}\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma<0}}\gamma(x_0)^{-m_\gamma} \quad\text{and}\quad \pm\,\alpha(y)^m\alpha(x)^{m_\alpha}\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma>0}}\gamma(x_0)^{m_\gamma}\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma<0}}\gamma(y_0)^{-m_\gamma}$$

respectively and for m_α negative, the terms of lowest degree in $(\alpha(x),\alpha(y))$ of left and right sides are

$$\pm\alpha(x)^{m+m_\alpha}\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma>0}}\gamma(y_0)^{m_\gamma}\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma<0}}\gamma(x_0)^{-m_\gamma} \quad\text{and}\quad \pm\,\alpha(y)^{m+m_\alpha}\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma>0}}\gamma(x_0)^{m_\gamma}\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma<0}}\gamma(y_0)^{-m_\gamma}$$

respectively. From the equality of these terms, we deduce $m=\pm m_\alpha$ and

$$\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma>0}}\gamma(y_0)^{m_\gamma}\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma<0}}\gamma(x_0)^{-m_\gamma} = \pm\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma>0}}\gamma(x_0)^{m_\gamma}\prod_{\substack{\gamma\in\mathcal{R}_+\setminus\{\alpha\}\\ m_\gamma<0}}\gamma(y_0)^{-m_\gamma}.$$

Since the last equality does not depend on the choice of subregular elements x_0 and y_0 in $\mathfrak{h}_\alpha$, this equality remains true for all (x_0,y_0) in $\mathfrak{h}_\alpha\times\mathfrak{h}_\alpha$. As a result, as the degrees in $\alpha(x)$ of the left and right sides of Equality (2) are the same,

$$m - \sum_{\substack{\gamma\in\mathcal{R}_+\\ m_\gamma<0 \text{ and } \gamma(H_\alpha)\neq 0}} m_\gamma = \sum_{\substack{\gamma\in\mathcal{R}_+\\ m_\gamma>0 \text{ and } \gamma(H_\alpha)\neq 0}} m_\gamma. \tag{3}$$

Suppose $m=m_\alpha$. By Equality (1),

$$\prod_{\gamma\in\mathcal{R}_+\setminus\{\alpha\}}\gamma(x)^{m_\gamma}\gamma(y)^{-m_\gamma} = \pm 1.$$

Since this equality does not depend on the choice of the subregular elements x_0 and y_0 in $\mathfrak{h}_\alpha$, it holds for all (x,y) in $\mathfrak{h}_{\rm reg}\times\mathfrak{h}_{\rm reg}$. Hence $m_\gamma=0$ for all γ in $\mathcal{R}_+\setminus\{\alpha\}$ and $m=0$ by Equality (3). It is impossible since $m_\alpha\neq 0$. Hence $m=-m_\alpha$. Then, by Equality (1)

$$\prod_{\gamma\in\mathcal{R}_+\setminus\{\alpha\}}\gamma(x)^{m_\gamma}\gamma(y)^{-m_\gamma} = \pm\alpha(x)^{2m}\alpha(y)^{-2m}.$$

Since this equality does not depend on the choice of the subregular elements x_0 and y_0 in $\mathfrak{h}_\alpha$, it holds for all (x, y) in $\mathfrak{h}_{\rm reg} \times \mathfrak{h}_{\rm reg}$. Then $m = 0$, whence the contradiction. □

4.4. For $2 \leqslant k' \leqslant k$, the projection

$$\mathfrak{g}^k \longrightarrow \mathfrak{g}^{k'}, \qquad (x_1, \ldots, x_k) \longmapsto (x_1, \ldots, x_{k'})$$

induces the projections

$$\mathcal{C}^{(k)} \longrightarrow \mathcal{C}^{(k')}, \qquad V_j^{(k)} \longrightarrow V_j^{(k')}.$$

Set:

$$V_{1,2}^{(k)} := V_1^{(k)} \cap V_2^{(k)}.$$

Lemma 4.9. *Let ω be a regular differential form of top degree on $V_1^{(k)}$, without zero. Denote by ω' its restriction to $V_{1,2}^{(k)}$.*

(i) *For φ in $\Bbbk[V_1^{(k)}]$, if φ has no zero then φ is in $\Bbbk^*$.*

(ii) *For some invertible element ψ of $\Bbbk[V_{1,2}^{(2)}]$, $\omega' = \psi\sigma_{2*}(\omega')$.*

(iii) *The function $\psi(\psi\circ\sigma_2)$ on $V_{1,2}^{(k)}$ is equal to 1.*

Proof. Following the arguments of the proof of Lemma 4.5, the lemma results from Lemma 4.2,(ii). □

Corollary 4.10. *The function ψ is invariant under the action of G in $V_{1,2}^{(k)}$ and for some sequence $m_\alpha, \alpha \in \mathcal{R}_+$ in $\mathbb{Z}$,*

$$\psi(x_1, \ldots, x_k) = \pm \prod_{\alpha\in\mathcal{R}_+} (\alpha(x_1)\alpha(x_2)^{-1})^{m_\alpha},$$

for all $(x_1, \ldots, x_k)$ in $\mathfrak{h}_{\rm reg}^2 \times \mathfrak{h}^k$.

Proof. The corollary results from Lemma 4.9 by the arguments of the proof of Corollary 4.6. □

Proposition 4.11. *There exists on $V^{(k)}$ a regular differential form of top degree without zero.*

Proof. As in the proof of Proposition 4.8, it suffices to prove that $m_\alpha = 0$ for all α in $\mathcal{R}_+$ since $G.(\mathfrak{h}_{\rm reg}^2 \times \mathfrak{h}^{k-2})$ is a dense open subset of $V^{(k)}$. As $V_{0,1,2}^{(k)}$ is contained in $V_{1,2}^{(k)}$, $m_\alpha = 0$ by the proof of Proposition 4.8. □

4.5. For $2 \leqslant k' \leqslant k$, the projection

$$\mathcal{X}^k \longrightarrow \mathcal{X}^{k'}, \qquad (x_1, \ldots, x_k, y_1, \ldots, y_k) \longmapsto (x_1, \ldots, x_{k'}, y_1, \ldots, y_{k'})$$

induces the projections

$$\mathcal{C}_{\rm x}^{(k)} \longrightarrow \mathcal{C}_{\rm x}^{(k')}, \qquad W_j^{(k)} \longrightarrow W_j^{(k')}.$$

Set:

$$W_{1,2}^{(k)} := W_1^{(k)} \cap W_2^{(k)}.$$

According to Corollary 4.3,(iii), $W_{1,2}^{(k)}$ is equal to $G.\iota_{{\rm x},k}(V_{0,1,2}^{(k)})$.

Lemma 4.12. *Let ω be a regular differential form of top degree on $W_1^{(k)}$, without zero. Denote by ω' its restriction to $W_{1,2}^{(k)}$.*

(i) *For φ in $\Bbbk[W_1^{(k)}]$, if φ has no zero then φ is in $\Bbbk^*$.*

(ii) *For some invertible element ψ of $\Bbbk[W_{1,2}^{(2)}]$, $\omega' = \psi\sigma_{2*}(\omega')$.*

(iii) *The function $\psi(\psi\circ\sigma_2)$ on $W_{1,2}^{(k)}$ is equal to 1.*

Proof. Following the arguments of the proof of Lemma 4.5, the lemma results from Lemma 4.2,(iii). $\square$

Corollary 4.13. *The function ψ is invariant under the action of G in $W^{(k)}_{1,2}$ and for some sequence $m_\alpha, \alpha \in \mathcal{R}_+$ in $\mathbb{Z}$,*

$$\psi\circ\iota_{\mathrm{x},k}(x_1,\dots,x_k) = \pm \prod_{\alpha\in\mathcal{R}_+} (\alpha(x_1)\alpha(x_2)^{-1})^{m_\alpha},$$

for all $(x_1,\dots,x_k)$ in $\mathfrak{h}^2_{\mathrm{reg}}\times\mathfrak{h}^k$.

Proof. Since $W^{(k)}_{1,2} = G.\iota_{\mathrm{x},k}(V^{(k)}_{0,1,2})$, the corollary results from Lemma 4.12 by the arguments of the proof of Corollary 4.6. $\square$

Proposition 4.14. *There exists on $W^{(k)}$ a regular differential form of top degree without zero.*

Proof. As in the proof of Proposition 4.8, it suffices to prove that $m_\alpha = 0$ for all α in $\mathcal{R}_+$ since $G.\iota_{\mathrm{x},k}(\mathfrak{h}^2_{\mathrm{reg}}\times\mathfrak{h}^{k-2})$ is a dense open subset of $W^{(k)}$. As $W^{(k)}_{1,2} = G.\iota_{\mathrm{x},k}(V^{(k)}_{0,1,2})$, $m_\alpha = 0$ by the proof of Proposition 4.8. $\square$

4.6. Recall that $(G.X)_{\mathrm{n}}$ is the normalization of $G.X$. Denote by $\mathcal{E}^{(k)}_{\mathrm{n}}$ the following fiber products:

$$\begin{tikzcd} \mathcal{E}^{(k)}_{\mathrm{n}} \arrow[r, "\nu_{\mathrm{n},k}"] \arrow[d, "\pi_{\mathrm{n},k}"'] & \mathcal{E}^{(k)} \arrow[d, "\pi"] \\ (G.X)_{\mathrm{n}} \arrow[r, "\nu"'] & G.X \end{tikzcd}$$

with ν the normalization morphism, $\pi_{\mathrm{n},k}$, $\nu_{\mathrm{n},k}$ the restriction maps.

Lemma 4.15. *The variety $\mathcal{E}^{(k)}_{\mathrm{n}}$ is the normalization of $\mathcal{E}^{(k)}$ and $\nu_{\mathrm{n},k}$ is the normalization morphism.*

Proof. Since $\mathcal{E}^{(k)}$ is a vector bundle over $G.X$, $\mathcal{E}^{(k)}_{\mathrm{n}}$ is a vector bundle over $(G.X)_{\mathrm{n}}$. Then $\mathcal{E}^{(k)}_{\mathrm{n}}$ is normal since so is $(G.X)_{\mathrm{n}}$. Moreover, the fields of rational functions on $\mathcal{E}^{(k)}_{\mathrm{n}}$ and $\mathcal{E}^{(k)}$ are equal and the comorphism of $\nu_{\mathrm{n},k}$ induces the morphism identity of this field so that $\nu_{\mathrm{n},k}$ is the normalization morphism. $\square$

Denote by $\widetilde{\mathfrak{X}_{0,k}}$, $\widetilde{\mathcal{C}^{(k)}}$, $\widetilde{\mathcal{C}^{(k)}_{\mathrm{x}}}$ the normalizations of $\mathfrak{X}_{0,k}$, $\mathcal{C}^{(k)}$, $\mathcal{C}^{(k)}_{\mathrm{x}}$ respectively. Let $\lambda_{0,k}$, λ_k, $\lambda_{*,k}$ be the respective normalization morphisms.

Lemma 4.16. (i) *There exists a projective birational morphism $\tau_{\mathrm{n},0,k}$ from $\mathcal{E}^{(k)}_0$ onto $\widetilde{\mathfrak{X}_{0,k}}$ such that $\tau_{0,k} = \lambda_{0,k}\circ\tau_{\mathrm{n},0,k}$. Moreover, $\tau^{-1}_{\mathrm{n},0,k}(\lambda^{-1}_{0,k}(V^{(k)}_0))$ is a smooth big open subset of $\mathcal{E}^{(k)}_0$ and the restriction of $\tau_{\mathrm{n},0,k}$ to this subset is an isomorphism onto $\lambda^{-1}_{0,k}(V^{(k)}_0)$.*

(ii) *There exists a projective birational morphism $\tau_{\mathrm{n},k}$ from $\mathcal{E}^{(k)}_{\mathrm{n}}$ onto $\widetilde{\mathcal{C}^{(k)}}$ such that $\tau_k\circ\nu_{\mathrm{n},k} = \lambda_k\circ\tau_{\mathrm{n},k}$. Moreover, $\tau^{-1}_{\mathrm{n},k}(\lambda^{-1}_k(V^{(k)}))$ is a smooth big open subset of $\mathcal{E}^{(k)}_{\mathrm{n}}$ and the restriction of $\tau_{\mathrm{n},k}$ to this subset is an isomorphism onto $\lambda^{-1}_k(V^{(k)})$.*

(iii) *There exists a projective birational morphism $\tau_{\mathrm{n},*,k}$ from $G\times_B\mathcal{E}^{(k)}_0$ onto $\widetilde{\mathcal{C}^{(k)}_{\mathrm{x}}}$ such that $\tau_{*,k} = \lambda_{*,k}\circ\tau_{\mathrm{n},*,k}$. Moreover, $\tau^{-1}_{\mathrm{n},*,k}(\lambda^{-1}_{*,k}(W^{(k)}))$ is a smooth big open subset of $G\times_B\mathcal{E}^{(k)}_0$ and the restriction of $\tau_{\mathrm{n},*,k}$ to this subset is an isomorphism onto $\lambda^{-1}_{*,k}(W^{(k)})$.*

Proof. (i) According to Corollary 4.3,(i), $\tau_{0,k}$ is a birational morphism from $\mathcal{E}^{(k)}_0$ onto $\mathfrak{X}_{0,k}$ and $\mathcal{E}^{(k)}_0$ is a normal variety since so is X by [C15, Theorem 1.1]. Hence it factorizes through $\lambda_{0,k}$ so that for some

birational morphism $\tau_{n,0,k}$ from $\mathcal{E}_0^{(k)}$ to $\widetilde{\mathfrak{X}_{0,k}}$, $\tau_{0,k} = \lambda_{0,k} \circ \tau_{n,0,k}$, whence the commutative diagram:

$$\begin{tikzcd} & \mathcal{E}_0^{(k)} \arrow[dl, "\tau_{n,0,k}"'] \arrow[dr, "\tau_{0,k}"] & \\ \widetilde{\mathfrak{X}_{0,k}} \arrow[rr, "\lambda_{0,k}"'] & & \mathfrak{X}_{0,k} \end{tikzcd} .$$

According to Lemma 4.1,(i), $\tau_{0,k}$ is a projective morphism. Hence so is $\tau_{n,0,k}$ since it deduces from $\tau_{0,k}$ by base extension [H77, Ch. II, Exercise 4.9].

According to Lemma 4.4,(i), $\tau_{0,k}^{-1}(V_0^{(k)})$ is a big open subset of $\mathcal{E}_0^{(k)}$. Moreover, we have the commutative diagram

$$\begin{tikzcd} & \tau_{0,k}^{-1}(V_0^{(k)}) \arrow[dl, "\tau_{n,0,k}"'] \arrow[dr, "\tau_{0,k}"] & \\ \lambda_{0,k}^{-1}(V_0^{(k)}) \arrow[rr, "\lambda_{0,k}"'] & & V_0^{(k)} \end{tikzcd} .$$

By Lemma 4.2,(i), $V_0^{(k)}$ is a smooth open subset of $\mathfrak{X}_{0,k}$ so that $\lambda_{0,k}$ is an isomorphism from $\lambda_{0,k}^{-1}(V_0^{(k)})$ onto $V_0^{(k)}$. By Corollary 4.3,(i), $\tau_{0,k}$ is an isomorphism from $\tau_{0,k}^{-1}(V_0^{(k)})$ onto $V_0^{(k)}$ so that $\tau_{0,k}^{-1}(V_0^{(k)})$ is a smooth open subset of $\mathcal{E}_0^{(k)}$. As a result, $\tau_{n,0,k}$ is an isomorphism from $\tau_{0,k}^{-1}(V_0^{(k)})$ onto $\lambda_{0,k}^{-1}(V_0^{(k)})$.

(ii) According to Corollary 4.3,(ii), $\tau_k \circ \nu_{n,k}$ is a birational morphism from $\mathcal{E}_n^{(k)}$ onto $\mathcal{C}^{(k)}$ and $\mathcal{E}_n^{(k)}$ is a normal variety by Lemma 4.15,(i). Hence it factorizes through λ_k so that for some birational morphism $\tau_{n,k}$ from $\mathcal{E}_n^{(k)}$ to $\widetilde{\mathcal{C}^{(k)}}$, $\tau_k \circ \nu_{n,k} = \lambda_k \circ \tau_{n,k}$, whence the commutative diagram:

$$\begin{tikzcd} \mathcal{E}_n^{(k)} \arrow[r, "\nu_{n,k}"] \arrow[d, "\tau_{n,k}"'] & \mathcal{E}^{(k)} \arrow[d, "\tau_k"] \\ \widetilde{\mathcal{C}^{(k)}} \arrow[r, "\lambda_k"'] & \mathcal{C}^{(k)} \end{tikzcd} .$$

According to Lemma 4.1,(i), τ_k is a projective morphism. Hence so is $\tau_{n,k}$ since it deduces from τ_k by base extension [H77, Ch. II, Exercise 4.9].

According to Lemma 4.4,(ii), $\tau_{n,k}^{-1}(\lambda_k^{-1}(V^{(k)}))$ is a big open subset of $\mathcal{E}_n^{(k)}$ since $\nu_{n,k}$ is a finite morphism. Moreover, we have the commutative diagram

$$\begin{tikzcd} \tau_{n,k}^{-1}(\lambda_k^{-1}(V^{(k)})) \arrow[r, "\nu_{n,k}"] \arrow[d, "\tau_{n,k}"'] & \tau_k^{-1}(V^{(k)}) \arrow[d, "\tau_k"] \\ \lambda_k^{-1}(V^{(k)}) \arrow[r, "\lambda_k"'] & V^{(k)} \end{tikzcd} .$$

By Lemma 4.2,(ii), $V^{(k)}$ is a smooth open subset of $\mathcal{C}^{(k)}$ so that λ_k is an isomorphism from $\lambda_k^{-1}(V^{(k)})$ onto $V^{(k)}$. By Corollary 4.3,(ii), τ_k is an isomorphism from $\tau_k^{-1}(V^{(k)})$ onto $V^{(k)}$ so that $\tau_k^{-1}(V^{(k)})$ is a smooth open subset of $\mathcal{E}^{(k)}$ and $\nu_{n,k}$ is an isomorphism from $\tau_{n,k}^{-1}(\lambda_k^{-1}(V^{(k)}))$ onto $\tau_k^{-1}(V^{(k)})$. As a result, $\tau_{n,k}$ is an isomorphism from $\tau_{n,k}^{-1}(\lambda_k^{-1}(V^{(k)}))$ onto $\lambda_k^{-1}(V^{(k)})$ and $\tau_{n,k}^{-1}(\lambda_k^{-1}(V^{(k)}))$ is a smooth open subset of $\mathcal{E}_n^{(k)}$.

(iii) According to Corollary 4.3,(iii), $\tau_{*,k}$ is a birational morphism from $G \times_B \mathcal{E}_0^{(k)}$ onto $\mathcal{C}_x^{(k)}$ and $G \times_B \mathcal{E}_0^{(k)}$ is a normal variety as a vector bundle over $G \times_B X$ which is normal by Proposition 3.2. Hence it factorizes

through $\lambda_{*,k}$ so that for some birational morphism $\tau_{\mathrm{n},*,k}$ from $G\times_B \mathcal{E}_0^{(k)}$ to $\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}$, $\tau_{*,k} = \lambda_{*,k}\circ\tau_{\mathrm{n},*,k}$, whence the commutative diagram:

$$\begin{tikzcd} & G\times_B \mathcal{E}_0^{(k)} \arrow[dl, "\tau_{\mathrm{n},*,k}"'] \arrow[dr, "\tau_{*,k}"] & \\ \widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}} \arrow[rr, "\lambda_{*,k}"'] & & \mathcal{C}_{\mathrm{x}}^{(k)} \end{tikzcd} .$$

According to Lemma 4.1,(i), $\tau_{*,k}$ is a projective morphism. Hence so is $\tau_{\mathrm{n},*,k}$ since it deduces from $\tau_{*,k}$ by base extension [H77, Ch. II, Exercise 4.9].

According to Lemma 4.4,(iii), $\tau_{*,k}^{-1}(W^{(k)})$ is a big open subset of $G\times_B \mathcal{E}_0^{(k)}$. Moreover, we have the commutative diagram

$$\begin{tikzcd} & \tau_{*,k}^{-1}(W^{(k)}) \arrow[dl, "\tau_{\mathrm{n},*,k}"'] \arrow[dr, "\tau_{*,k}"] & \\ \lambda_{*,k}^{-1}(W^{(k)}) \arrow[rr, "\lambda_{*,k}"'] & & W^{(k)} \end{tikzcd} .$$

By Lemma 4.2,(iii), $W^{(k)}$ is a smooth open subset of $\mathcal{C}_{\mathrm{x}}^{(k)}$ so that $\lambda_{*,k}$ is an isomorphism from $\lambda_{*,k}^{-1}(W^{(k)})$ onto $W^{(k)}$. By Corollary 4.3,(i), $\tau_{*,k}$ is an isomorphism from $\tau_{*,k}^{-1}(W^{(k)})$ onto $W^{(k)}$ so that $\tau_{*,k}^{-1}(W^{(k)})$ is a smooth open subset of $G\times_B \mathcal{E}_0^{(k)}$. As a result, $\tau_{\mathrm{n},*,k}$ is an isomorphism from $\tau_{*,k}^{-1}(W^{(k)})$ onto $\lambda_{*,k}^{-1}(W^{(k)})$. □

Let $\mathfrak{Y}$ be one of the three varieties $\widetilde{\mathfrak{X}_{0,k}}$, $\widetilde{\mathcal{C}^{(k)}}$, $\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}$ and set:

$$\mathfrak{Z} := \begin{cases} \mathcal{E}_0^{(k)} & \text{if} \quad \mathfrak{Y} = \widetilde{\mathfrak{X}_{0,k}} \\ \mathcal{E}_{\mathrm{n}}^{(k)} & \text{if} \quad \mathfrak{Y} = \widetilde{\mathcal{C}^{(k)}} \\ G\times_B \mathcal{E}_0^{(k)} & \text{if} \quad \mathfrak{Y} = \widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}} \end{cases}, \qquad \tau := \begin{cases} \tau_{\mathrm{n},0,k} & \text{if} \quad \mathfrak{Y} = \widetilde{\mathfrak{X}_{0,k}} \\ \tau_{\mathrm{n},k} & \text{if} \quad \mathfrak{Y} = \widetilde{\mathcal{C}^{(k)}} \\ \tau_{\mathrm{n},*,k} & \text{if} \quad \mathfrak{Y} = \widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}} \end{cases},$$

$$\mathfrak{T} := \begin{cases} X & \text{if} \quad \mathfrak{Y} = \widetilde{\mathfrak{X}_{0,k}} \\ G.X_{\mathrm{n}} & \text{if} \quad \mathfrak{Y} = \widetilde{\mathcal{C}^{(k)}} \\ G\times_B X & \text{if} \quad \mathfrak{Y} = \widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}} \end{cases}, \qquad \pi := \begin{cases} \mathcal{E}_0^{(k)} \longrightarrow X & \text{if} \quad \mathfrak{Y} = \widetilde{\mathfrak{X}_{0,k}} \\ \mathcal{E}_{\mathrm{n}}^{(k)} \longrightarrow (G.X)_{\mathrm{n}} & \text{if} \quad \mathfrak{Y} = \widetilde{\mathcal{C}^{(k)}} \\ G\times_B \mathcal{E}_{\mathrm{n}}^{(k)} \longrightarrow G\times_B X & \text{if} \quad \mathfrak{Y} = \widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}} \end{cases},$$

where the arrow is the bundle projection in the last three equalities.

Proposition 4.17. (i) *The morphism τ is a projective birational morphism.*

(ii) *The set $\mathfrak{Z}_{\mathrm{sm}}$ is the inverse image of $\mathfrak{T}_{\mathrm{sm}}$ by π.*

(iii) *For some smooth big open subset $\mathfrak{O}$ of $\mathfrak{Z}_{\mathrm{sm}}$, the restriction of τ to $\mathfrak{O}$ is an isomorphism onto a smooth big open subset of $\mathfrak{Y}$.*

(iv) *The sheaves $\Omega_{\mathfrak{Y}_{\mathrm{sm}}}$ and $\Omega_{\mathfrak{Z}_{\mathrm{sm}}}$ have a global section without* 0.

Proof. (i) The assertion results from Lemma 4.16.

(ii) As a polynomial algebra over an algebra A is regular if and only if so is A, $\mathfrak{Z}_{\mathrm{sm}} = \pi^{-1}(\mathfrak{T}_{\mathrm{sm}})$ since $\mathfrak{Z}$ is a vector bundle over $\mathfrak{T}$.

(iii) The assertion results from Lemma 4.16 and Lemma 4.4.

(iv) For $\mathfrak{Y} = \widetilde{\mathfrak{X}_{0,k}}$, the assertion results from Lemma C.1, Proposition 4.8, Lemma 4.2,(i) and Lemma 4.4,(i). For $\mathfrak{Y} = \widetilde{\mathcal{C}^{(k)}}$, the assertion results from Lemma C.1, Proposition 4.11, Lemma 4.2,(ii) and Lemma 4.4,(ii).

For $\mathfrak{Y} = \widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}$, the assertion results from Lemma C.1, Proposition 4.14, Lemma 4.2,(iii) and Lemma 4.4,(iii). □

5. Rational singularities

Let $k \geqslant 2$ be an integer and let $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{T}$, τ, π be as in Proposition 4.17. Denote by ι the canonical embeddings $\mathfrak{Y}_{\mathrm{sm}} \longrightarrow \mathfrak{Y}$ and $\mathfrak{Z}_{\mathrm{sm}} \longrightarrow \mathfrak{Z}$. According to [Hir64], there exists a desingularization Γ of $\mathfrak{T}$ with morphism θ such that the restriction of θ to $\theta^{-1}(\mathfrak{T}_{\mathrm{sm}})$ is an isomorphism onto $\mathfrak{T}_{\mathrm{sm}}$. Let $\widetilde{\mathfrak{Z}}$ be the following fiber product

$$\begin{tikzcd} \widetilde{\mathfrak{Z}} \arrow[r, "\overline{\theta}"] \arrow[d, "\overline{\pi}"'] & \mathfrak{Z} \arrow[d, "\pi"] \\ \Gamma \arrow[r, "\theta"'] & \mathfrak{T} \end{tikzcd}$$

with $\overline{\theta}$ and $\overline{\pi}$ the restriction maps so that $\widetilde{\mathfrak{Z}}$ is a vector bundle of rank $k\ell$ over Γ and $\overline{\pi}$ is the bundle projection. Moreover, $\overline{\theta}$ is projective and birational so that $\widetilde{\mathfrak{Z}}$ is a desingularization of $\mathfrak{Z}$ and $\mathfrak{Y}$ by Proposition 4.17,(i).

Proposition 5.1. *Suppose $\mathfrak{Y} = \widetilde{\mathfrak{X}_{0,k}}$ or $\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}$.*

(i) *The variety $\mathfrak{Z}$ is Gorenstein with rational singularities. Moreover, its canonical bundle is free of rank one.*

(ii) *The variety $\mathfrak{Y}$ is Gorenstein with rational singularities. Moreover, its canonical bundle is free of rank one.*

Proof. (i) According to [C15, Theorem 1.1], X has rational singularities. Then, by Lemma D.1,(iii), so has $G \times_B X$ as a fiber bundle over a smooth variety with fibers having rational singularities. As a result, by Lemma D.1,(iv), $\mathfrak{Z}$ has rational singularities as a vector bundle over a variety having rational singularities. Moreover, $\mathfrak{T}$ is Gorenstein by Proposition 3.2,(i) and (iii). Then so is $\mathfrak{Z}$ as a vector bundle over $\mathfrak{T}$ by Lemma D.1,(i). By Proposition 4.17, $\Omega_{\mathfrak{Z}_{\mathrm{sm}}}$ has a global section without zero. Then, by Lemma C.2, $\iota_*(\Omega_{\mathfrak{Z}_{\mathrm{sm}}})$ is a free module of rank one. Since $\mathfrak{Z}$ has rational singularities, the canonical module of $\mathfrak{Z}$ is equal to $\iota_*(\Omega_{\mathfrak{Z}_{\mathrm{sm}}})$ by [KK73, p.50], whence the assertion.

(ii) By Proposition 4.17, $\Omega_{\mathfrak{Y}_{\mathrm{sm}}}$ has a global section without zero. Denote it by ω. By Proposition 4.17,(iii), $\tau^*(\omega)$ is a local section of $\Omega_{\mathfrak{Z}_{\mathrm{sm}}}$ above a big open subset of $\mathfrak{Z}$. So by (i) and [KK73, p.50], $\tau^*(\omega)$ has a regular extension to $\widetilde{\mathfrak{Z}}$. Denote it by $\widetilde{\omega}$ and by μ the morphism

$$\mathcal{O}_{\widetilde{\mathfrak{Z}}} \xrightarrow{\ \mu\ } \Omega_{\widetilde{\mathfrak{Z}}} , \qquad \varphi \longmapsto \varphi\widetilde{\omega}.$$

Since ω has no zero, by Lemma C.2, $\tau\circ\overline{\theta}_*\mu$ is an isomorphism from $\mathcal{O}_{\mathfrak{Y}}$ onto $\Omega_{\mathfrak{Y}_{\mathrm{sm}}}$ and $\iota_*(\Omega_{\mathfrak{Y}_{\mathrm{sm}}})$ is a free module of rank one. As a result, by [Hi91, Lemma 2.3], $\mathfrak{Y}$ is Gorenstein with rational singularities. Then, by [KK73, p.50], the canonical module of $\mathfrak{Y}$ is equal to $\iota_*(\Omega_{\mathfrak{Y}_{\mathrm{sm}}})$, whence the assertion. □

Corollary 5.2. (i) *The variety $\widetilde{\mathcal{C}^{(k)}}$ is Gorenstein with rational singularities. Moreover its canonical module is free of rank one.*

(ii) *The variety $\mathcal{E}_{\mathrm{n}}^{(k)}$ is Gorenstein with rational singularities. Moreover its canonical module is free of rank one.*

(iii) *The varieties $\mathcal{E}_{\mathrm{n}}$ and $(G.X)_{\mathrm{n}}$ are Gorenstein with rational singularities.*

Proof. In the proof, we suppose $\mathfrak{Y} = \widetilde{\mathcal{C}^{(k)}}$.

(i) According to [CZ14, Proposition 5.8,(ii)], $\widetilde{\mathcal{C}^{(k)}}$ is the categorical quotient of $\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}$ by the action of $W(\mathcal{R})$. Hence, by [Boutot87, Théorème] and Proposition 5.1,(ii), $\mathfrak{Y} = \widetilde{\mathcal{C}^{(k)}}$ has rational singularities. By Proposition 4.17,(iv), $\Omega_{\mathfrak{Y}_{\mathrm{sm}}}$ has a global section without zero. Then, by Lemma C.2, $\iota_*(\Omega_{\mathfrak{Y}_{\mathrm{sm}}})$ is a free module of rank one. Since $\mathfrak{Y}$ has rational singularities, the canonical module of $\mathfrak{Y}$ is equal to $\iota_*(\Omega_{\mathfrak{Z}_{\mathrm{sm}}})$ by [KK73, p.50]. Moreover, $\mathfrak{Y}$ is Cohen-Macaulay. So, by Lemma C.3, $\iota_*(\Omega_{\mathfrak{Z}_{\mathrm{sm}}})$ has finite injective dimension, whence $\mathfrak{Y}$ is Gorenstein.

(ii) Denote by ω a global section of $\Omega_{\mathfrak{Y}_{\mathrm{sm}}}$ without zero. By Proposition 4.17,(iii) and Lemma C.2, $\Omega_{\mathfrak{Z}_{\mathrm{sm}}}$ has a global section without zero whose restriction to a big open subset of $\mathfrak{Z}_{\mathrm{sm}}$ is equal to the restriction of $\tau^*(\omega)$. Denote it by ω'. Since $\mathfrak{Y}$ has rational singularities, $\tau{\circ}\overline{\theta}^*(\omega)$ has a regular extension to $\widetilde{\mathfrak{Z}}$ by [KK73, p.50]. Denote it by $\widetilde{\omega}$. Then the restriction of $\widetilde{\omega}$ to $\overline{\theta}^{-1}(\mathfrak{Z}_{\mathrm{sm}})$ is equal to $\overline{\theta}^*(\omega')$. Let μ be the morphism

$$\mathcal{O}_{\widetilde{\mathfrak{Z}}} \xrightarrow{\ \mu\ } \Omega_{\widetilde{\mathfrak{Z}}}\ , \qquad \varphi \longmapsto \varphi\widetilde{\omega}.$$

Since ω' has no zero, by Lemma C.2, $\overline{\theta}_*\mu$ is an isomorphism from $\mathcal{O}_{\mathfrak{Z}}$ onto $\Omega_{\mathfrak{Z}_{\mathrm{sm}}}$ and $\iota_*(\Omega_{\mathfrak{Z}_{\mathrm{sm}}})$ is free of rank one. As a result, by [Hi91, Lemma 2.3], $\mathfrak{Z}$ is Gorenstein with rational singularities. Then, by [KK73, p.50], the canonical module of $\mathfrak{Z}$ is equal to $\iota_*(\Omega_{\mathfrak{Z}_{\mathrm{sm}}})$, whence the assertion.

(iii) Since $\mathfrak{Z}$ is a vector bundle over $\mathfrak{T} = (G.X)_{\mathrm{n}}$, $(G.X)_{\mathrm{n}}$ is Gorenstein with rational singularities by (ii) and Lemma D.1,(ii) and (iv). Then so is $\mathcal{E}_{\mathrm{n}}$ as a vector bundle over $(G.X)_{\mathrm{n}}$ by Lemma D.1,(i) and (iv). □

Summarizing the results, Theorem 1.1 results from Proposition 5.1,(ii), and Corollary 5.2,(i) and (iii). According to [Ri79], $\mathcal{C}^{(2)}$ is the commuting variety of $\mathfrak{g}$ and according to [C12, Theorem 1.1], $\mathcal{C}^{(2)}$ is normal, whence:

Corollary 5.3. *The commuting variety of* $\mathfrak{g}$ *is Gorenstein with rational singularities. Moreover, its canonical module is free of rank* 1.

6. Normality

Let k be a positive integer. The goal of this section is to prove that $\mathfrak{X}_{0,k}$ is a normal variety. Consider the desingularization (Γ, θ) of X as in Section 5. For simplicity of the notations, for k positive integer, we denote by π_k the bundle projection $\mathcal{E}_0^{(k)} \longrightarrow X$ and by $F^{(k)}$ the fiber product

$$\begin{array}{ccc} F^{(k)} & \xrightarrow{\ \theta_k\ } & \mathcal{E}_0^{(k)} \\ {\scriptstyle \overline{\pi_k}}\downarrow & & \downarrow{\scriptstyle \pi_k} \\ \Gamma & \xrightarrow[\ \theta\ ]{} & X \end{array}$$

with θ_k and $\overline{\pi_k}$ the restriction morphisms.

6.1. Let F^* be the dual of the vector bundle $F^{(1)}$ over Γ.

Lemma 6.1. *Let* $\mathcal{F}^*$ *be the sheaf of local sections of* F^**. For* $i > 0$ *and for* $j \geqslant 0$, $\mathrm{H}^i(\Gamma, \mathrm{S}^j(\mathcal{F}^*)) = 0$.

Proof. Since $\overline{\pi_1}$ is the bundle projection of the vector bundle $F^{(1)}$ over Γ, $\mathcal{O}_{F^{(1)}}$ is equal to $\overline{\pi_1}^*(\mathrm{S}(\mathcal{F}^*))$ so that

$$\overline{\pi_1}_*(\mathcal{O}_F) = \mathrm{S}(\mathcal{F}^*)$$

As a result, for $i \geqslant 0$,

$$\mathrm{H}^i(F^{(1)}, \mathcal{O}_{F^{(1)}}) = \mathrm{H}^i(\Gamma, \mathrm{S}(\mathcal{F}^*)) = \bigoplus_{j\in\mathbb{N}} \mathrm{H}^i(\Gamma, \mathrm{S}^j(\mathcal{F}^*))$$

According to Lemma 3.1,(i), $F^{(1)}$ is a desingularization of the smooth variety $\mathfrak{b}$. Hence by [El78],

$$\mathrm{H}^i(F^{(1)}, \mathcal{O}_{F^{(1)}}) = 0$$

for $i > 0$, whence

$$\mathrm{H}^i(\Gamma, \mathrm{S}^j(\mathcal{F}^*)) = 0$$

for $i > 0$ and $j \geqslant 0$. □

According to the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ by the bilinear form $\langle .\,,.\rangle$, $\mathfrak{b}_-$ identifies with $\mathfrak{b}^*$. Denote by F_- the orthogonal complement to $F^{(1)}$ in $\Gamma \times \mathfrak{b}_-$ so that F_- is a vector bundle of rank n over Γ. Let $\mathcal{F}_-$ be the sheaf of local sections of F_-.

Corollary 6.2. *Let $\mathcal{J}_0$ be the ideal of $\mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)$ generated by $\mathcal{F}_-$. Then, for $i \geqslant 0$, $\mathrm{H}^i(\Gamma, \mathcal{J}_0) = 0$ and $\mathrm{H}^i(\Gamma, \mathcal{F}_-) = 0$.*

Proof. Since F_- is the orthogonal complement to $F^{(1)}$ in $\Gamma \times \mathfrak{b}_-$, $\mathcal{J}_0$ is the ideal of definition of $F^{(1)}$ in $\mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)$ whence a short exact sequence

$$0 \longrightarrow \mathcal{J}_0 \longrightarrow \mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-) \longrightarrow \mathrm{S}(\mathcal{F}^*) \longrightarrow 0$$

and whence a cohomology long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(\Gamma, \mathrm{S}(\mathcal{F}^*)) \longrightarrow \mathrm{H}^{i+1}(\Gamma, \mathcal{J}_0) \longrightarrow \mathrm{H}^{i+1}(\Gamma, \mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)) \longrightarrow \cdots .$$

Then, by Lemma 6.1, from the equality

$$\mathrm{H}^i(\Gamma, \mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)) = \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathrm{H}^i(\Gamma, \mathcal{O}_\Gamma)$$

for all i, we deduce $\mathrm{H}^i(\Gamma, \mathcal{J}_0) = 0$ for $i \geqslant 2$. Moreover, since Γ is an irreducible projective variety, $\mathrm{H}^0(\Gamma, \mathcal{O}_\Gamma) = \Bbbk$ and since $F^{(1)}$ is a desingularization of $\mathfrak{b}$, $\mathrm{H}^0(\Gamma, \mathrm{S}(\mathcal{F}^*)) = \mathrm{S}(\mathfrak{b}_-)$ so that the map

$$\mathrm{H}^0(\Gamma, \mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)) \longrightarrow \mathrm{H}^0(\Gamma, \mathrm{S}(\mathcal{F}^*))$$

is an isomorphism. Hence $\mathrm{H}^i(\Gamma, \mathcal{J}_0) = 0$ for $i = 0, 1$. The gradation on $\mathrm{S}(\mathfrak{b}_-)$ induces a gradation on $\mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)$ so that $\mathcal{J}_0$ is a graded ideal. Since $\mathcal{F}_-$ is the subsheaf of local sections of degree 1 of $\mathcal{J}_0$, it is a direct factor of $\mathcal{J}_0$, whence the corollary. □

Proposition 6.3. *Let k, l be nonnegative integers.*

(i) *For all positive integer i, $\mathrm{H}^i(\Gamma, (\mathcal{F}^*)^{\otimes k}) = 0$.*

(ii) *For all positive integer i,*

$$\mathrm{H}^{i+l}(\Gamma, \mathcal{F}_-^{\otimes l} \otimes_{\mathcal{O}_\Gamma} (\mathcal{F}^*)^{\otimes k}) = 0.$$

Proof. (i) According to Lemma 6.1, we can suppose $k > 1$. Denote by F_k^* the restriction to the diagonal of Γ^k of the vector bundle F^{*k} over Γ^k. Identifying Γ with the diagonal of Γ^k, F_k^* is a vector bundle over Γ. Since F^* is the dual of the vector bundle $F^{(1)}$ over Γ, F_k^* is the dual of the vector bundle $F^{(k)}$ over Γ. Let ψ_k be the bundle projection of F_k^* and let $\mathcal{F}_k^*$ be the sheaf of local sections of F_k^*. Then $\mathcal{O}_{F^{(k)}}$ is equal to $\psi_k^*(\mathrm{S}(\mathcal{F}_k^*))$ and since $F^{(k)}$ is a vector bundle over Γ, for all nonnegative integer i,

$$\mathrm{H}^i(F^{(k)}, \mathcal{O}_{F^{(k)}}) = \mathrm{H}^i(\Gamma, \mathrm{S}(\mathcal{F}_k^*)) = \bigoplus_{q\in\mathbb{N}} \mathrm{H}^i(\Gamma, \mathrm{S}^q(\mathcal{F}_k^*)).$$

According to Proposition 5.1,(ii), for $i > 0$, the left hand side is equal to 0 since $F^{(k)}$ is a desingularization of $\widetilde{\mathfrak{X}_{0,k}}$ by Proposition 4.17,(i). As a result, for $i > 0$,

$$\mathrm{H}^i(\Gamma, \mathrm{S}^k(\mathcal{F}_k^*)) = 0.$$

The decomposition of $\mathcal{F}_k^*$ as a direct sum of k copies isomorphic to $\mathcal{F}^*$ induces a multigradation of $\mathrm{S}(\mathcal{F}_k^*)$. Denoting by $\mathcal{S}_{j_1,\ldots,j_k}$ the subsheaf of multidegree $(j_1, \ldots, j_k)$, we have

$$\mathrm{S}^k(\mathcal{F}_k^*) = \bigoplus_{\substack{(j_1,\ldots,j_k)\in\mathbb{N}^k \\ j_1+\cdots+j_k=k}} \mathcal{S}_{j_1,\ldots,j_k} \quad \text{and} \quad \mathcal{S}_{1,\ldots,1} = (\mathcal{F}^*)^{\otimes k}.$$

Hence for $i > 0$,

$$0 = \mathrm{H}^i(\Gamma, \mathrm{S}^k(\mathcal{S}_k^*)) = \bigoplus_{\substack{(j_1,\ldots,j_k)\in\mathbb{N}^k \\ j_1+\cdots+j_k=k}} \mathrm{H}^i(\Gamma, \mathcal{S}_{j_1,\ldots,j_k})$$

whence the assertion.

(ii) Let k be a nonnegative integer. Prove by induction on j that for $i > 0$ and for $l \geqslant j$,

$$\mathrm{H}^{i+j}(\Gamma, \mathcal{F}_-^{\otimes j} \otimes_{\mathcal{O}_\Gamma} (\mathcal{F}^*)^{\otimes(k+l-j)}) = 0. \tag{4}$$

By (i) it is true for $j = 0$. Suppose $j > 0$ and (4) true for $j-1$ and for all $l \geqslant j-1$. From the short exact sequence of $\mathcal{O}_\Gamma$-modules

$$0 \longrightarrow \mathcal{F}_- \longrightarrow \mathcal{O}_\Gamma \otimes_{\Bbbk} \mathfrak{b}_- \longrightarrow \mathcal{F}^* \longrightarrow 0$$

we deduce the short exact sequence of $\mathcal{O}_\Gamma$-modules

$$0 \longrightarrow \mathcal{F}_-^{\otimes j} \otimes_{\mathcal{O}_\Gamma} (\mathcal{F}^*)^{\otimes(k+l-j)} \longrightarrow \mathfrak{b}_- \otimes_{\Bbbk} \mathcal{F}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{F}^*)^{\otimes(k+l-j)} \longrightarrow \mathcal{F}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{F}^*)^{\otimes(k+l-j+1)} \longrightarrow 0.$$

From the cohomology long exact sequence deduced from this short exact sequence, we have the exact sequence

$$\begin{aligned}\mathrm{H}^{i+j-1}(\Gamma, \mathcal{F}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{F}^*)^{\otimes(k+l-j+1)}) &\longrightarrow \mathrm{H}^{i+j}(\Gamma, \mathcal{F}_-^{\otimes j} \otimes_{\mathcal{O}_\Gamma} (\mathcal{F}^*)^{\otimes(k+l-j)}) \\ &\longrightarrow \mathrm{H}^{i+j}(\Gamma, \mathfrak{b}_- \otimes_{\Bbbk} \mathcal{F}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{F}^*)^{\otimes(k+l-j)})\end{aligned}$$

for all positive integer i. By induction hypothesis, the first term equals 0 for all $i > 0$. Since

$$\mathrm{H}^{i+j}(\Gamma, \mathfrak{b}_- \otimes_{\Bbbk} \mathcal{F}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{F}^*)^{\otimes(k+l-j)}) = \mathfrak{b}_- \otimes_{\Bbbk} \mathrm{H}^{i+j}(\Gamma, \mathcal{F}_-^{\otimes(j-1)} \otimes_{\mathcal{O}_\Gamma} (\mathcal{F}^*)^{\otimes(k+l-j)}),$$

the last term of the last exact sequence equals 0 by induction hypothesis again, whence Equality (4) and whence the assertion for $j = l$. □

The following corollary results from Proposition 6.3,(ii) and Proposition B.1.

Corollary 6.4. *For k positive integer and for $l = (l_1, \ldots, l_k)$ in $\mathbb{N}^k$,*

$$\mathrm{H}^{i+|l|}(\Gamma, \bigwedge^{l_1} \mathcal{F}_- \otimes_{\mathcal{O}_\Gamma} \cdots \otimes_{\mathcal{O}_\Gamma} \bigwedge^{l_k} \mathcal{F}_-) = 0$$

for all positive integer i.

6.2. By definition, $F^{(k)}$ is a closed subvariety of $\Gamma\times\mathfrak{b}^k$. Denote by ϱ the canonical projection from $\Gamma\times\mathfrak{b}^k$ to Γ, whence the diagram

$$\begin{tikzcd} F^{(k)} \arrow[r, hook] \arrow[dr, "\overline{\pi_k}"'] & \Gamma\times\mathfrak{b}^k \arrow[d, "\varrho"] \\ & \Gamma \end{tikzcd}$$

For $j=1,\dots,k$, denote by $\mathfrak{S}_{j,k}$ the set of injections from $\{1,\dots,j\}$ to $\{1,\dots,k\}$ and for σ in $\mathfrak{S}_{j,k}$, set:

$$\mathcal{K}_\sigma := \mathcal{M}_1\otimes_{\mathcal{O}_\Gamma}\cdots\otimes_{\mathcal{O}_\Gamma}\mathcal{M}_k \text{ with } \mathcal{M}_i := \left\{\begin{array}{ll} \mathcal{O}_\Gamma\otimes_{\Bbbk}\mathrm{S}(\mathfrak{b}_-) & \text{if} \quad i\notin\sigma(\{1,\dots,j\}) \\ \mathcal{J}_0 & \text{if} \quad i\in\sigma(\{1,\dots,j\}) \end{array}\right.$$

For j in $\{1,\dots,k\}$, the direct sum of the $\mathcal{K}_\sigma$'s is denoted by $\mathcal{J}_{j,k}$ and for σ in $\mathfrak{S}_{1,k}$, $\mathcal{K}_\sigma$ is also denoted by $\mathcal{K}_{\sigma(1),k}$.

Lemma 6.5. *Let $\mathcal{J}$ be the ideal of definition of $F^{(k)}$ in $\mathcal{O}_{\Gamma\times\mathfrak{b}^k}$.*

(i) *The ideal $\varrho_*(\mathcal{J})$ of $\mathcal{O}_\Gamma\otimes_{\Bbbk}\mathrm{S}(\mathfrak{b}_-^k)$ is the sum of $\mathcal{K}_{1,k},\dots,\mathcal{K}_{k,k}$.*

(ii) *There is an exact sequence of $\mathcal{O}_\Gamma$-modules*

$$0\longrightarrow\mathcal{J}_{k,k}\longrightarrow\mathcal{J}_{k-1,k}\longrightarrow\cdots\longrightarrow\mathcal{J}_{1,k}\longrightarrow\varrho_*(\mathcal{J})\longrightarrow 0$$

(iii) *For $i>0$, $\mathrm{H}^i(\Gamma\times\mathfrak{b}^k,\mathcal{J})=0$ if $\mathrm{H}^{i+j}(\Gamma,\mathcal{J}_0^{\otimes j})=0$ for $j=1,\dots,k$.*

Proof. (i) Let $\mathcal{J}_k$ be the sum of $\mathcal{K}_{1,k},\dots,\mathcal{K}_{k,k}$. Since $\mathcal{J}_0$ is the ideal of $\mathcal{O}_\Gamma\otimes_{\Bbbk}\mathrm{S}(\mathfrak{b}_-)$ generated by $\mathcal{F}_-$, $\mathcal{J}_k$ is a prime ideal of $\mathcal{O}_\Gamma\otimes_{\Bbbk}\mathrm{S}(\mathfrak{b}_-^k)$. Moreover, $\mathcal{F}_-$ is the sheaf of local sections of the orthogonal complement to F in $\Gamma\times\mathfrak{b}_-$. Hence $\mathcal{J}_k$ is the ideal of definition of $F^{(k)}$ in $\mathcal{O}_\Gamma\otimes_{\Bbbk}\mathrm{S}(\mathfrak{b}_-^k)$, whence the assertion.

(ii) For a a local section of $\mathcal{J}_{j,k}$ and for σ in $\mathfrak{S}_{j,k}$, denote by $a_{\sigma(1),\dots,\sigma(j)}$ the component of a on $\mathcal{K}_\sigma$. Let d be the map $\mathcal{J}_{j,k}\rightarrow\mathcal{J}_{j-1,k}$ such that

$$\mathrm{d}a_{i_1,\dots,i_j} = \sum_{l=1}^{j}(-1)^{l+1}a_{i_1,\dots,i_{l-1},i_{l+1},\dots,i_j}$$

Then by (i), we have an augmented complex

$$0\longrightarrow\mathcal{J}_{k,k}\xrightarrow{\mathrm{d}}\mathcal{J}_{k-1,k}\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}\mathcal{J}_{1,k}\longrightarrow\varrho_*(\mathcal{J})\longrightarrow 0.$$

Let J be the subbundle of the trivial bundle $\Gamma\times\mathrm{S}(\mathfrak{b}_-)$ such that the fiber at x is the ideal of $\mathrm{S}(\mathfrak{b}_-)$ generated by the fiber $F_{-,x}$ of F_- at x. Then $\mathcal{J}_0$ is the sheaf of local sections of J and the above augmented complex is the sheaf of local sections of the augmented complex of vector bundles over Γ,

$$0\longrightarrow C_k^{(k)}(\Gamma\times\mathrm{S}(\mathfrak{b}_-),J)\longrightarrow\cdots\longrightarrow C_1^{(k)}(\Gamma\times\mathrm{S}(\mathfrak{b}_-),J)\rightarrow J\longrightarrow 0$$

defined as in Subsection B.2. According to Lemma B.2 and Remark B.3, this complex is acyclic, whence the assertion by Nakayama Lemma since J, $\mathrm{S}(\mathfrak{b}_-)$ and the complex are graded.

(iii) Let i be a positive integer such that $\mathrm{H}^{i+j}(\Gamma,\mathcal{J}_0^{\otimes j})=0$ for $j=1,\dots,k$. Then for $j=1,\dots,k$ and for σ in $\mathfrak{S}_{j,k}$, $\mathrm{H}^{i+j}(\Gamma,\mathcal{K}_\sigma)=0$ since $\mathcal{K}_\sigma$ is isomorphic to a sum of copies of $\mathcal{J}_0^{\otimes j}$. Moreover, $\mathrm{H}^i(\Gamma,\mathcal{K}_{l,k})=0$ for $l=1,\dots,k$ since $\mathrm{H}^i(\Gamma,\mathcal{J}_0)=0$ by Corollary 6.2. Hence by (ii), since $\mathrm{H}^\bullet$ is an exact δ-functor, $\mathrm{H}^i(\Gamma,\varrho_*(\mathcal{J}))=0$, whence the assertion since ϱ is an affine morphism. □

6.3. For k positive integer, for j nonnegative integer and for $l = (l_1, \dots, l_k)$ in $\mathbb{N}^k$, set:

$$\mathcal{M}_{j,l} := \mathcal{J}_0^{\otimes j} \otimes_{\mathcal{O}_\Gamma} \bigwedge^{l_1} \mathcal{F}_- \otimes_{\mathcal{O}_\Gamma} \cdots \otimes_{\mathcal{O}_\Gamma} \bigwedge^{l_k} \mathcal{F}_-$$

Lemma 6.6. *Let k be a positive integer and let (j,l) be in $\mathbb{N} \times \mathbb{N}^k$.*

(i) *The $\mathcal{O}_\Gamma$-module $\mathcal{J}_0$ is locally free.*

(ii) *For $j > 0$, there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathcal{M}_{j-1,(n,l)} \longrightarrow \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathcal{M}_{j-1,(n-1,l)} \longrightarrow \cdots \\ \longrightarrow \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathcal{M}_{j-1,(1,l)} \longrightarrow \mathcal{M}_{j,l} \longrightarrow 0 \end{aligned}$$

(iii) *For $i > 0$, $\mathrm{H}^{i+j+|l|}(\Gamma, \mathcal{M}_{j,l}) = 0$.*

Proof. (i) Let x be in Γ and let $F_{-,x}$ be the fiber at x of the vector bundle F_- over Γ. Then $F_{-,x}$ is a subspace of dimension n of $\mathfrak{b}_-$. Let M be a complement to $F_{-,x}$ in $\mathfrak{b}_-$. Since the map $y \mapsto F_{-,y}$ is a regular map from Γ to $\mathrm{Gr}_n(\mathfrak{b}_-)$, for all y in an open neighborhood V of x in Γ,

$$\mathfrak{b}_- = F_{-,x} \oplus M$$

Denoting by $\mathcal{F}_{-,V}$ the restriction of $\mathcal{F}_-$ to V, we have

$$\mathcal{O}_V \otimes_{\Bbbk} \mathfrak{b}_- = \mathcal{F}_{-,V} \oplus \mathcal{O}_V \otimes_{\Bbbk} M$$

so that

$$\mathcal{O}_V \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-) = \mathrm{S}(\mathcal{F}_{-,V}) \otimes_{\Bbbk} \mathrm{S}(M)$$

whence

$$\mathcal{J}_0 \,|_V = \mathrm{S}_+(\mathcal{F}_{-,V}) \otimes_{\Bbbk} \mathrm{S}(M).$$

As a result, $\mathcal{J}_0$ is locally free since so is $\mathcal{F}_-$.

(ii) Since $\mathcal{J}_0$ is the ideal of $\mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)$ generated by the locally free module $\mathcal{F}_-$ of rank n and since $\mathcal{F}_-$ is locally generated by a regular sequence of the algebra $\mathcal{O}_\Gamma \otimes_{\Bbbk} \mathrm{S}(\mathfrak{b}_-)$, having n elements, we have an exact Koszul complex

$$0 \longrightarrow \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \bigwedge^{n} \mathcal{F}_- \longrightarrow \cdots \longrightarrow \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathcal{F}_- \longrightarrow \mathcal{J}_0 \longrightarrow 0$$

whence a complex

$$\begin{aligned} 0 \longrightarrow \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \bigwedge^{n} \mathcal{F}_- \otimes_{\mathcal{O}_\Gamma} \mathcal{M}_{j-1,l} \longrightarrow \cdots \longrightarrow \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathcal{F}_- \otimes_{\mathcal{O}_\Gamma} \mathcal{M}_{j-1,l} \\ \longrightarrow \mathcal{J}_0 \otimes_{\mathcal{O}_\Gamma} \mathcal{M}_{j-1,l} \longrightarrow 0. \end{aligned}$$

According to (i), $\mathcal{M}_{j-1,l}$ is a locally free module. Hence this complex is acyclic.

(iii) Prove the assertion by induction on j. According to Corollary 6.4, it is true for $j = 0$. Suppose that it is true for $j-1$. According to the induction hypothesis, for all positive integer i and for $p = 1, \dots, n$,

$$\mathrm{H}^{i+j-1+p+|l|}(\Gamma, \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathcal{M}_{j-1,(p,l)}) = \mathrm{S}(\mathfrak{b}_-) \otimes_{\Bbbk} \mathrm{H}^{i+j-1+p+|l|}(\Gamma, \mathcal{M}_{j-1,(p,l)}) = 0.$$

Then, according to (ii), $\mathrm{H}^{i+j+|l|}(\Gamma, \mathcal{M}_{j,l}) = 0$ for all positive integer i since $\mathrm{H}^\bullet$ is an exact δ-functor. □

Proposition 6.7. *The variety $\mathfrak{X}_{0,k}$ is Gorenstein with rational singularities and its canonical module is free of rank* 1. *Moreover the ideal of definition of $\mathfrak{X}_{0,k}$ in $\mathrm{S}(\mathfrak{b}_-)^{\otimes k}$ is the space of global sections of $\mathcal{J}$.*

Proof. From the short exact sequence,

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{\Gamma\times\mathfrak{b}^k} \longrightarrow \mathcal{O}_{F^{(k)}} \longrightarrow 0$$

we deduce the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(\Gamma\times\mathfrak{b}^k,\mathcal{J}) \longrightarrow \mathrm{S}(\mathfrak{b}_-)^{\otimes k}\otimes_{\Bbbk}\mathrm{H}^i(\Gamma,\mathcal{O}_\Gamma) \longrightarrow \mathrm{H}^i(F^{(k)},\mathcal{O}_{F^{(k)}}) \longrightarrow \mathrm{H}^{i+1}(\Gamma\times\mathfrak{b}^k,\mathcal{J}) \longrightarrow \cdots$$

According to Proposition 2.2,(i), $\mathrm{H}^i(\Gamma,\mathcal{O}_\Gamma)=0$ for $i>0$ and according to Lemma 6.5,(iii) and Lemma 6.6,(iii), $\mathrm{H}^i(\Gamma\times\mathfrak{b}^k,\mathcal{J})=0$ for $i>0$. Hence, $\mathrm{H}^i(F^{(k)},\mathcal{O}_{F^{(k)}})=0$ for $i>0$, whence the short exact sequence

$$0 \longrightarrow \mathrm{H}^0(\Gamma\times\mathfrak{b}^k,\mathcal{J}) \longrightarrow \mathrm{S}(\mathfrak{b}_-)^{\otimes k} \longrightarrow \mathrm{H}^0(F^{(k)},\mathcal{O}_{F^{(k)}}) \longrightarrow 0$$

As $F^{(k)}$ is a desingularization of $\mathfrak{X}_{0,k}$, $\Bbbk[\widetilde{\mathfrak{X}_{0,k}}]$ is the space of global sections of $\mathcal{O}_{F^{(k)}}$ by Lemma 1.2. Then $\Bbbk[\mathfrak{X}_{0,k}]=\Bbbk[\widetilde{\mathfrak{X}_{0,k}}]$ since the image of $\mathrm{S}(\mathfrak{b}_-)^{\otimes k}$ is contained in $\Bbbk[\mathfrak{X}_{0,k}]$, whence the proposition by Proposition 5.1,(ii). □

Corollary 6.8. (i) *The normalization morphism of* $\mathcal{C}_{\mathrm{x}}^{(k)}$ *is a homeomorphism.*

(ii) *The normalization morphism of* $\mathcal{C}^{(k)}$ *is a homeomorphism.*

Proof. (i) As $\mathfrak{X}_{0,k}$ is contained in $\mathfrak{b}^k$, we deduce the commutative diagram

$$\begin{tikzcd} G\times_B\mathfrak{X}_{0,k} \arrow[r, hook] \arrow[d] & G\times_B\mathfrak{b}^k \arrow[d, "\gamma_{\mathrm{x}}"] \\ \mathcal{C}_{\mathrm{x}}^{(k)} \arrow[r, hook] & \mathcal{B}_{\mathrm{x}}^{(k)} \end{tikzcd}$$

According to [CZ14, Proposition 3.4], the normalization morphism of $\mathcal{B}_{\mathrm{x}}^{(k)}$ is a homeomorphism. Then since $G\times_B\mathfrak{b}^k$ is a desingularization of $\mathcal{B}_{\mathrm{x}}^{(k)}$, the fibers of γ_{x} are connected by Zariski Main Theorem [Mu88, §9]. Then so are the fibers of the restriction of γ_{x} to $G\times_B\mathfrak{X}_{0,k}$ since $G\times_B\mathfrak{X}_{0,k}$ is the inverse image of $\mathcal{C}_{\mathrm{x}}^{(k)}$. According to Proposition 6.7, $G\times_B\mathfrak{X}_{0,k}$ is a normal variety. Moreover, the restriction of γ_{x} to $G\times_B\mathfrak{X}_{0,k}$ is projective and birational, whence the commutative diagram

$$\begin{tikzcd} G\times_B\mathfrak{X}_{0,k} \arrow[rr, "\widetilde{\gamma}_{\mathrm{x}}"] \arrow[dr, "\gamma_{\mathrm{x}}"'] & & \widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}} \arrow[dl, "\lambda_{*,k}"] \\ & \mathcal{C}_{\mathrm{x}}^{(k)} & \end{tikzcd}$$

with $\lambda_{*,k}$ the normalization morphism. For x in $\mathcal{C}_{\mathrm{x}}^{(k)}$, $\lambda_{*,k}^{-1}(x)=\widetilde{\gamma}_{\mathrm{x}}(\gamma_{\mathrm{x}}^{-1}(x))$. Hence $\lambda_{*,k}$ is injective since the fibers of γ_{x} are connected, whence the assertion since $\lambda_{*,k}$ is closed as a finite morphism.

(ii) Denote again by η the restriction of η to $\mathcal{C}_{\mathrm{x}}^{(k)}$. We have a commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}} \arrow[r, "\lambda_{*,k}"] \arrow[d, "\widetilde{\eta}"'] & \mathcal{C}_{\mathrm{x}}^{(k)} \arrow[d, "\eta"] \\ \widetilde{\mathcal{C}^{(k)}} \arrow[r, "\lambda_k"'] & \mathcal{C}^{(k)} \end{tikzcd}$$

with λ_k the normalization morphism. According to [CZ14, Proposition 5.8], all fiber of η or $\widetilde{\eta}$ is one single $W(\mathcal{R})$-orbit and by (i), $\lambda_{*,k}$ is bijective. Hence λ_k is bijective, whence the assertion since λ_k is closed as a finite morphism. □

Appendix A. Notations

In this appendix, V is a finite dimensional vector space. Denote by $\mathrm{S}(V)$ and $\bigwedge V$ respectively the symmetric and exterior algebras of V. For all integer i, $\mathrm{S}^i(V)$ and $\bigwedge^i V$ are the subspaces of degree i for the usual gradation of $\mathrm{S}(V)$ and $\bigwedge V$ respectively. In particular, $\mathrm{S}^i(V)$ and $\bigwedge^i V$ are equal to zero for i negative.

- For l positive integer, denote by $\mathfrak{S}_l$ the group of permutations of l elements.
- For m positive integer and for $l = (l_1, \dots, l_m)$ in $\mathbb{N}^m$, set:

$$\begin{aligned} |l| :=& \quad l_1 + \cdots + l_m \\ \mathrm{S}^l(V) :=& \quad \mathrm{S}^{l_1}(V) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathrm{S}^{l_m}(V) \\ \bigwedge^l V :=& \quad \bigwedge^{l_1} V \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \bigwedge^{l_m} V. \end{aligned}$$

- For k positive integer and for $l = (l_1, \dots, l_m)$ in $\mathbb{N}^m$ such that $1 \leqslant |l| \leqslant k$, denote by $V^{\otimes k}$ the k-th tensor power of V and by $\mathfrak{S}_l$ the direct product $\mathfrak{S}_{l_1} \times \cdots \times \mathfrak{S}_{l_m}$. The group $\mathfrak{S}_l$ has a natural action on $V^{\otimes k}$ given by

$$\begin{aligned} (\sigma_1, \dots, \sigma_m).(v_1 \otimes \cdots \otimes v_k) = & \quad v_{\sigma_1(1)} \otimes \cdots \otimes v_{\sigma_1(l_1)} \otimes v_{l_1+\sigma_2(1)} \otimes \cdots \otimes v_{l_1+\sigma_2(l_2)} \\ & \otimes \cdots \otimes v_{|l|-l_m+\sigma_m(1)} \otimes \cdots \otimes v_{|l|-l_m+\sigma_m(l_m)} \otimes v_{|l|+1} \otimes \cdots \otimes v_k. \end{aligned}$$

The map

$$a \longmapsto \pi_{k,l}(a) := \prod_{j=1}^{m} \frac{1}{l_j!} \sum_{\sigma \in \mathfrak{S}_l} \sigma.a$$

is a projection from $V^{\otimes k}$ onto $(V^{\otimes k})^{\mathfrak{S}_l}$. Moreover, the restriction to $(V^{\otimes k})^{\mathfrak{S}_l}$ of the canonical map from $V^{\otimes k}$ to $\mathrm{S}^l(V) \otimes_{\Bbbk} V^{\otimes (k-|l|)}$ is an isomorphism of vector spaces.

Appendix B. Some complexes

Let X be a smooth algebraic variety. For $\mathcal{M}$ a coherent $\mathcal{O}_X$-module and for k positive integer, denote by $\mathcal{M}^{\otimes k}$ the k-th tensor power of $\mathcal{M}$. According to Notations A, for all l in $\mathbb{N}^m$ such that $|l| \leqslant k$, there is an action of $\mathfrak{S}_l$ on $\mathcal{M}^{\otimes k}$. Moreover, $\mathrm{S}^l(\mathcal{M})$ and $\bigwedge^l \mathcal{M}$ are coherent modules defined by the same formulas as in Notations A.

B.1. Let $\mathcal{E}$ and $\mathcal{M}$ be locally free $\mathcal{O}_X$-modules.

Proposition B.1. *Let i be a positive integer and suppose that*

$$\mathrm{H}^{i+j}(X, \mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M}) = 0$$

for all nonnegative integers j, k.

(i) *For all positive integers m and k and for all l in $\mathbb{N}^m$ such that $|l| \leqslant k$,*

$$\mathrm{H}^i(X, \mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-|l|)} \otimes_{\mathcal{O}_X} \mathcal{M}) = 0.$$

(ii) *For all positive integers n_1, n_2, k and for all (l, m) in $\mathbb{N}^{n_1} \times \mathbb{N}^{n_2}$ such that $|l| + |m| \leqslant k$,*

$$\mathrm{H}^i(X, \mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \bigwedge^m \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-|l|-|m|)} \otimes_{\mathcal{O}_X} \mathcal{M}) = 0.$$

Proof. (i) Since $\pi_{k,l}(\mathcal{E}^{\otimes k})$ is isomorphic to $\mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|)}$ and since $\pi_{k,l}$ is a projector of $\mathcal{E}^{\otimes k}$, $\mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|)}$ is isomorphic to a direct factor of $\mathcal{E}^{\otimes k}$ and $\mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|)} \otimes_{\mathcal{O}_X} \mathcal{M}$ is isomorphic to a direct factor of $\mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M}$, whence the assertion.

(ii) Denoting by $\varepsilon(\sigma)$ the signature of the element σ of the symmetric group $\mathfrak{S}_m$, the map

$$\mathcal{E}^{\otimes m} \longrightarrow \mathcal{E}^{\otimes m} \qquad a \mapsto \tfrac{1}{m!} \textstyle\sum_{\sigma \in \mathfrak{S}_m} \varepsilon(\sigma)\sigma.a$$

is a projection from $\mathcal{E}^{\otimes m}$ onto a submodule of $\mathcal{E}^{\otimes m}$ isomorphic to $\bigwedge^m \mathcal{E}$. So, $\bigwedge^m \mathcal{E}$ is isomorphic to a direct factor of $\mathcal{E}^{\otimes m}$. Then, by induction on m, for l in $\mathbb{N}^m$, $\bigwedge^l \mathcal{E}$ is isomorphic to a direct factor of $\mathcal{E}^{\otimes |l|}$. As a result, according to (i), for all positive integers n_1, n_2, k and for all (l, m) in $\mathbb{N}^{n_1} \times \mathbb{N}^{n_2}$ such that $|l| + |m| \leqslant k$, $\mathrm{S}^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \bigwedge^m \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes(k-|l|-|m|)} \otimes_{\mathcal{O}_X} \mathcal{M}$ is isomorphic to a direct factor of $\mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M}$, whence the assertion. □

B.2. Let W be a subspace of V and set $E := V/W$. Let $C_{\bullet}^{(n)}(V, W)$, $n = 1, 2, \ldots$ be the sequence of graded spaces over $\mathbb{N}$ defined by the induction relations:

$$C_0^{(1)}(V, W) := V \qquad C_1^{(1)}(V, W) := W \qquad C_i^{(1)}(V, W) := 0$$
$$C_0^{(n)}(V, W) := V^{\otimes n} \qquad C_j^{(n)}(V, W) := C_j^{(n-1)}(V, W) \otimes_{\Bbbk} V \oplus C_{j-1}^{(n-1)}(V, W) \otimes_{\Bbbk} W$$

for $i \geqslant 2$ and $j \geqslant 1$.

Lemma B.2. *Let n be a positive integer. There exists a graded differential of degree -1 on $C_{\bullet}^{(n)}(V, W)$ such that the complex so defined has no homology in positive degree.*

Proof. Prove the lemma by induction on n. For $n = 1$, d is given by the inclusion map $W \longrightarrow V$. Suppose that $C_{\bullet}^{(n-1)}(V, W)$ has a differential d verifying the conditions of the lemma. For $j > 0$, denote by δ the linear map

$$C_j^{(n)}(V, W) \longrightarrow C_{j-1}^{(n)}(V, W), \qquad (a \otimes v, b \otimes w) \longmapsto (\mathrm{d}a \otimes v + (-1)^j b \otimes w, \mathrm{d}b \otimes w)$$

with a, b, v, w in $C_j^{(n-1)}(V, W)$, $C_{j-1}^{(n-1)}(V, W)$, V, W respectively. Then δ is a graded differential of degree -1. Let c be a cycle of positive degree j of $C_{\bullet}^{(n)}(V, W)$. Then c has an expansion

$$c = (\sum_{i=1}^{d} a_i \otimes v_i, \sum_{i=1}^{d'} b_i \otimes v_i)$$

with $v_1, \ldots, v_d$ a basis of V such that $v_1, \ldots, v_{d'}$ is a basis of W and with $a_1, \ldots, a_d$ and $b_1, \ldots, b_{d'}$ in $C_j^{(n-1)}(V, W)$ and $C_{j-1}^{(n-1)}(V, W)$ respectively. Since c is a cycle,

$$\sum_{i=1}^{d} \mathrm{d}a_i \otimes v_i + (-1)^j \sum_{i=1}^{d'} b_i \otimes v_i = 0$$

Hence $b_i = (-1)^{j+1} \mathrm{d}a_i$ for $i = 1, \ldots, d'$ so that

$$c + \delta(0, \sum_{i=1}^{d'} (-1)^j a_i \otimes v_i) = (\sum_{i=1}^{d} a_i \otimes v_i + \sum_{i=1}^{d'} a_i \otimes v_i, \sum_{i=1}^{d'} (b_i \otimes v_i + (-1)^j \mathrm{d}a_i \otimes v_i)) = (\sum_{i=1}^{d} a_i \otimes v_i + \sum_{i=1}^{d'} a_i \otimes v_i, 0).$$

So we can suppose $b_1 = \cdots = b_{d'} = 0$. Then $a_1, \ldots, a_d$ are cycles of degree j of $C_{\bullet}^{(n-1)}(V, W)$. By induction hypothesis, they are boundaries of $C_{\bullet}^{(n-1)}(V, W)$ so that c is a boundary of $C_{\bullet}^{(n)}(V, W)$, whence the lemma. □

Remark B.3. The results of this subsection remain true for V or W of infinite dimension since a vector space is an inductive limit of finite dimensional vector spaces.

Let X be an affine irreducible normal variety.

Lemma C.1. *Let Y be a smooth big open subset of X.*

(i) *All regular differential form of top degree on Y has a unique regular extension to $X_{\rm sm}$.*

(ii) *Suppose that ω is a regular differential form of top degree on Y, without zero. Then the regular extension of ω to $X_{\rm sm}$ has no zero.*

Proof. (i) Since $\Omega_{X_{\rm sm}}$ is a locally free module of rank one, there is an affine open cover $O_1,\ldots,O_k$ of $X_{\rm sm}$ such that the restriction of $\Omega_{X_{\rm sm}}$ to O_i is a free $\mathcal{O}_{O_i}$-module generated by some section ω_i. For $i=1,\ldots,k$, set $O'_i := O_i\cap Y$. Let ω be a regular differential form of top degree on Y. For $i=1,\ldots,k$, for some regular function a_i on O'_i, $a_i\omega_i$ is the restriction of ω to O'_i. As Y is a big open subset of X, O'_i is a big open subset of O_i. Hence a_i has a regular extension to O_i since O_i is normal. Denoting again by a_i this extension, for $1\leqslant i,j\leqslant k$, $a_i\omega_i$ and $a_j\omega_j$ have the same restriction to $O'_i\cap O'_j$ and $O_i\cap O_j$ since $\Omega_{X_{\rm sm}}$ is torsion free as a locally free module. Let ω' be the global section of $\Omega_{X_{\rm sm}}$ extending the $a_i\omega_i$'s. Then ω' is a regular extension of ω to $X_{\rm sm}$ and this extension is unique since Y is dense in $X_{\rm sm}$ and $\Omega_{X_{\rm sm}}$ is torsion free.

(ii) Suppose that ω has no zero. Let Σ be the nullvariety of ω' in $X_{\rm sm}$. If it is not empty, Σ has codimension 1 in $X_{\rm sm}$. As Y is a big open subset of X, $\Sigma\cap X_{\rm sm}$ is not empty if so is Σ. As a result, Σ is empty. $\square$

Denote by ι the inclusion morphism $X_{\rm sm} \longrightarrow X$.

Lemma C.2. *Suppose that $\Omega_{X_{\rm sm}}$ has a global section ω without zero. Then the $\mathcal{O}_X$-module $\iota_*(\Omega_{X_{\rm sm}})$ is free of rank 1. More precisely, the morphism θ:*

$$\mathcal{O}_X \xrightarrow{\ \theta\ } \iota_*(\Omega_{X_{\rm sm}})\ , \qquad \psi \longmapsto \psi\omega$$

is an isomorphism.

Proof. For φ a local section of $\iota_*(\Omega_{X_{\rm sm}})$ above the open subset U of X, for some regular function ψ on $U\cap X_{\rm sm}$,

$$\psi(\omega\big|_{U\cap X_{\rm sm}}) = \varphi.$$

Since X is normal, so is U and $U\cap X_{\rm sm}$ is a big open subset of U. Hence ψ has a regular extension to U. As a result, there exists a well defined morphism from $\iota_*(\Omega_{X_{\rm sm}})$ to $\mathcal{O}_X$ whose inverse is θ. $\square$

According to [Hir64], X has a desingularization Z with morphism τ such that the restriction of τ to $\tau^{-1}(X_{\rm sm})$ is an isomorphism onto $X_{\rm sm}$. Since Z and X are varieties over $\Bbbk$, we have the commutative diagram

$$\begin{tikzcd} Z \arrow[rr, "\tau"] \arrow[dr, "p"'] & & X \arrow[dl, "q"] \\ & \mathrm{Spec}(\Bbbk) & \end{tikzcd} .$$

According to [H66, V. §10.2], $p^!(\Bbbk)$ and $q^!(\Bbbk)$ are dualizing complexes over Z and X respectively. Furthermore, by [H66, VII, 3.4] or [Hi91, 4.3,(ii)], $p^!(\Bbbk)[-\dim Z]$ equals Ω_Z. Set $\mathcal{D} := q^!(\Bbbk)[-\dim Z]$ so that $\tau^!(\mathcal{D}) = \Omega_Z$ by [H66, VII, 3.4] or [Hi91, 4.3,(iv)]. In particular, $\mathcal{D}$ is dualizing over X.

Lemma C.3. *Suppose that X has rational singularities. Let $\mathcal{M}$ be the cohomology in degree 0 of $\mathcal{D}$. Then the $\mathcal{O}_X$-modules $\tau_*(\Omega_Z)$ and $\mathcal{M}$ are isomorphic. In particular, $\tau_*(\Omega_Z)$ has finite injective dimension.*

Proof. Since τ is a projective morphism, we have the isomorphism

$$\mathrm{R}\tau_*(\mathrm{R}\mathscr{H}om_Z(\Omega_Z, \Omega_Z)) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathrm{R}(\tau)_*(\Omega_Z), \mathcal{D}) \tag{5}$$

by [H66, VII, 3.4] or [Hi91, 4.3,(iii)]. Since $\mathrm{H}^i(\mathrm{R}\mathscr{H}om_Z(\Omega_Z, \Omega_Z)) = \mathcal{O}_Z$ for $i = 0$ and 0 for $i > 0$, the left hand side of (5) can be identified with $\mathrm{R}\tau_*(\mathcal{O}_Z)$. Since X has rational singularities, $\mathrm{R}\tau_*(\mathcal{O}_Z) = \mathcal{O}_X$ and $\mathcal{D}$ has only cohomology in degree. Moreover, by Grauert-Riemenschneider Theorem [GR70], $\mathrm{R}\tau_*(\Omega_Z)$ has only cohomology in degree 0, whence $R\tau_*(\Omega_Z) = \tau_*(\Omega_Z)$. Then, by (5), we have the isomorphism

$$\mathcal{O}_X \longrightarrow \mathscr{H}om_X((\tau)_*(\Omega_Z), \mathcal{M}) \ .$$

As $\mathcal{D}$ is dualizing, we have the isomorphism

$$R\tau_*(\Omega_Z) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathrm{R}\mathscr{H}om_X(R\tau_*(\Omega_Z), \mathcal{D}), \mathcal{D})$$

whence the isomorphism $\tau_*(\Omega_Z) \longrightarrow \mathcal{M}$. As a result, $\tau_*(\Omega_Z)$ has finite injective dimension since so has $\mathcal{M}$. ◻

Appendix D. About singularities

In this section we recall a well known result. Let X be a variety and Y a fiber bundle over X. Denote by τ the bundle projection.

Lemma D.1. (i) *If X is Gorenstein and the fibers of τ are Gorenstein, then so is Y.*

(ii) *If Y is a Gorenstein vector bundle over X, then X is Gorenstein.*

(iii) *Suppose that X and the fibers of τ have rational singularities. Then so has Y.*

(iv) *If Y is a vector bundle over X, X has rational singularities if and only if so has Y.*

Proof. Let y be in Y, $x := \tau(y)$ and F_x the fiber of Y at x. Denote by $\widehat{\mathcal{O}_{X,x}}$ and $\widehat{\mathcal{O}_{Y,y}}$ the completions of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ respectively.

(i) By hypothesis, $\mathcal{O}_{X,x}$ and $\mathcal{O}_{F_x,y}$ are Gorenstein. Then so is $\mathcal{O}_{X,x} \otimes_{\Bbbk} \mathcal{O}_{F_x,y}$. So by [Bru, Proposition 3.1.19,(a)], $\mathcal{O}_{Y,y}$ is Gorenstein, whence the assertion.

(ii) Since Y is a vector bundle over X, $\widehat{\mathcal{O}_{Y,y}}$ is a ring of formal series over $\widehat{\mathcal{O}_{X,x}}$. By [Bru, Proposition 3.1.19,(c)], $\widehat{\mathcal{O}_{Y,y}}$ is Gorenstein. So, by [Bru, Proposition 3.1.19,(b)], $\widehat{\mathcal{O}_{X,x}}$ is Gorenstein. Then by [Bru, Proposition 3.1.19,(c)], $\mathcal{O}_{X,x}$ is Gorenstein, whence the assertion.

(iii) There exists a cover of X by open subsets O such that $\tau^{-1}(O)$ is isomorphic to $O \times F$. According to the hypothesis, O and F have rational singularities. Then so has $\tau^{-1}(O)$, whence the assertion since a variety has rational singularities if and only it has a cover by open subsets having rational singularities.

(iv) If Y is a vector bundle over X, then there exists a cover of X by open subsets O, such that $\tau^{-1}(O)$ is isomorphic to $O \times \Bbbk^m$ with $m = \dim Y - \dim X$. According to [KK73, p.50], $O \times \Bbbk^m$ has rational singularities if and only if so has O, whence the assertion since a variety has rational singularities if and only it has a cover by open subsets having rational singularities. ◻

References

[Bol91] A.V. Bolsinov, *Commutative families of functions related to consistent Poisson brackets*, Acta Applicandae Mathematicae, **24** (1991), **n°1**, p. 253–274.

[BoK79] W. Borho and H. Kraft, *Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen*, Commentarii Mathematici Helvetici, **54** (1979), p. 61–104.

[Bou02] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6. Translated from the 1968 French original by Andrew Pressley*, Springer-Verlag, Berlin (2002).

[Bou98] N. Bourbaki, *Algèbre commutative, Chapitre 10, Éléments de mathématiques*, Masson (1998), Paris.

[Boutot87] J-François. Boutot, *Singularités rationnelles et quotients par les groupes réductifs*, Inventiones Mathematicae **88** (1987), p. 65–68.

[Bru] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics **n°39**, Cambridge University Press, Cambridge (1996).

[C12] J-Y. Charbonnel *On the Commuting variety of a reductive Lie algebra*, arXiv 1206.5592.

[C15] J-Y. Charbonnel *On a variety related to the commuting variety of a reductive Lie algebra.*, arXiv:1507.05419.

[CZ14] J-Y. Charbonnel and M. Zaiter *On the Commuting variety of a reductive Lie algebra and other related varieties*, arXiv 1204.0377.

[El78] R. Elkik, *Singularités rationnelles et déformations*, Inventiones Mathematicae **47** (1978), p. 139–147.

[El81] R. Elkik, *Rationalité des singularités canoniques*, Inventiones Mathematicae **64** (1981), p. 1–6.

[Go64] R. Godement, *Théorie des Faisceaux*, Actualités scientifiques et industrielles 1252, Publications de l'Institut Mathématique de l'université de Strasbourg XIII, Hermann, Paris.

[EGAII] A. Grothendieck *Éléments de Géométrie Algébrique II - Étude globale élémentaire de quelques classes de morphismes.* Publications Mathématiques **n°8** (1961), Le Bois-Marie, Bures sur Yvette.

[EGAIV] A. Grothendieck *Éléments de Géométrie Algébrique IV -Étude locale des schémas et des morphismes de schémas.* Publications Mathématiques **n°28–32** (1966-67), Le Bois-Marie, Bures sur Yvette.

[Gi12] V. Ginzburg *Isospectral commuting variety, the Harish-Chandra $\mathcal{D}$-module, and principal nilpotent pairs* Duke Mathematical Journal, **161** (2012), p. 2023–2111.

[GR70] H. Grauert, O. Riemenschneider *Verschwindungssätze für analytiscbe Kohomologie-gruppen auf komplexen Räumen* Inventiones Mathematicae **11** (1970), p. 263–292.

[H66] R. Hartshorne, *Residues and Duality*, Lecture Notes in Mathematics **n°20** (1966), Springer-Verlag, Berlin Heidelberg New York.

[H77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **n°52** (1977), Springer-Verlag, Berlin Heidelberg New York.

[Ha99] M. Haiman, *Macdonald polynomials and geometry. New perspectives in algebraic combinatorics (Berkeley, CA,1996–97)*, Math. Sci. Res. Inst. Publ. **38** (1999), Cambridge Univ. Press, Cambridge, p. 207–254.

[Ha02] M. Haiman, *Combinatorics, symmetric functions and Hilbert schemes*, Current Developments in Mathematics, **1** (2002), p. 39–111.

[Hi91] V. Hinich, *On the singularities of nilpotent orbits*, Israel Journal of Mathematice, **73** (1991), p. 297–308.

[Hir64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I,II*, Annals of Mathematics **79** (1964), p. 109–203 and p. 205–326.

[Hu95] James E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs, **n°43**, (1995).

[Ke77] G. R. Kempf, *Some quotient varieties have rational singularities*, Michigan Mathematical Journal **24** (1977), p. 347–352.

[KK73] G. R. Kempf, F. Knusdon, D. Mumford and B. Saint-Donat, *Toroidal embeddings*, Lecture Notes in Mathematics **n°339** (1973), Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.

[Ko63] B. Kostant, *Lie group representations on polynomial rings*, American Journal of Mathematics **85** (1963), p. 327–404.

[MA86] H. Matsumura, *Commutative ring theory* Cambridge studies in advanced mathematics **n°8** (1986), Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney.

[Mebk89] Z. Mebkhout, Systèmes différentiels, Le formalisme des six opérations de Grothendieck pour les D_X-modules cohérents, Travaux en cours, **35**, Hermann, Paris.

[MF78] A.S. Mishchenko and A.T. Fomenko, *Euler equations on Lie groups*, Math. USSR-Izv. **12** (1978), p. 371–389.

[Mu65] D. Mumford, *Geometric Invariant Theory*, Ergebnisse der Mathematik und Ihrer Grenzgzbiete **band 34** (1965), Springer-Verlag, Berlin, Heidelberg, New York.

[Mu88] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Mathematics **n°1358** (1988), Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.

[Po08] V.L. Popov *Irregular and singular loci of commuting varieties*, Transformation Groups **13** (2008), p. 819–837.

[Po08] V.L. Popov and E. B. Vinberg, *Invariant Theory, in: Algebraic Geometry IV*, Encyclopaedia of MathematicalSciences **n°55** (1994), Springer-Verlag, Berlin, p.123–284.

[Ri79] R. W. Richardson, *Commuting varieties of semisimple Lie algebras and algebraic groups*, Compositio Mathematica **38** (1979), p. 311–322.

[Sh94] I.R. Shafarevich, *Basic algebraic geometry 2*, Springer-Verlag (1994), Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong-Kong, Barcelona, Budapest.

[TY05] P. Tauvel and R.W.T. Yu, *Lie algebras and algebraic groups*, Monographs in Mathematics (2005), Springer, Berlin Heidelberg New York.

[ZS67] O. Zariski and P. Samuel, *Commutative Algebra*, D. Van Nostrand Company Incorporation (1967), Princeton, New Jersey, Toronto, New-York, London.

Jean-Yves Charbonnel, Université Paris Diderot - CNRS, Institut de Mathématiques de Jussieu - Paris Rive Gauche, UMR 7586, Groupes, représentations et géométrie, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France

E-mail address: `jean-yves.charbonnel@imj-prg.fr`

Mouchira Zaiter, Université Libanaise Al- Hadath, Faculté des sciences, branche I, Beyrouth Liban

E-mail address: `zaiter.mouchira@hotmail.fr`